

Principles of Finance (FINE 31123)

Class Notes

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1 Some fundamental concepts

Def: A **financial asset** is an asset that grants its owner the right to a stream of future cash flows noted $\{\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_N\}$.

These cash flows can be certain or uncertain. In the later case, each flow can be characterized by a probability distribution and the stream of cash flows by a joint probability distribution of all cash flows.

Examples of financial assets:

1. **Stocks (Equity):** Security granting an ownership right to the net asset of a corporation.

Shareholders are entitled to:

- dividends
- resale (market) price
- voting right

The stream of cash flows of a stock can be written as:

$$\{P_0, E(\tilde{D}_1), E(\tilde{D}_2), E(\tilde{D}_3), \dots, E(\tilde{D}_N), E(\tilde{P}_N)\}$$

where,

- P_0 : price of the stock today (t=0)
- $E(\tilde{D}_t)$: expected dividend at period t
- $E(\tilde{P}_N)$: expected price of the stock at period N

2. **Bonds:** Debt contract issued by governments and corporations.

Bondholders are entitled to:

- interests (fixed or variable)
- capital (principal)

The stream of cash flows of bond can be written as:

$$\{P_0, C_1, C_2, C_3, \dots, C_N, NV_N\}$$

where,

- P_0 : price of the bond today
- C_t : coupon (interests) paid at period t
- NV_N : nominal value (assumed to be equal to the principal) of the bond paid at period N
- In the case of a fixed interest rate “in fine” bond, the capital is entirely reimbursed at maturity and the bond pays constant interests (coupons) every year (or semester) to maturity.

3. **Derivative products:** Securities based on the value of an underlying asset. The three most common products are:

- options
- futures (standardized term contract and market traded)
- forwards (tailor-made term contract and over the counter traded)

Options will be covered later §8.

1.1 Value of a financial asset

Question: How can we determine the value of an asset?

Answer: An asset provides its owner the right to a stream of cash flows. Its value must then be equal to the present value of all cash flows generated by the asset.

The market price of an asset is determined by the supply and demand for the asset. We refer to it as the **equilibrium** price.

In equilibrium, one important condition must prevail: no arbitrage opportunities (absence of arbitrage).

Def: An **arbitrage opportunity** is a strategy that combines the purchases (long position) and the sales (short position) of an asset with i) no initial wealth and ii) no risk which provides a positive payoff with certainty.

Example: Consider the market for gold. Suppose that the price for an ounce of gold is \$3 800 in London and \$3 700 in New York. What is the right price?

- If the market for gold is competitive, this difference in price cannot prevail.
- An investor could short sell gold in London (receive \$3 800) and then immediately buy it in NY (pay \$3 700), thus ensuring a profit of \$100 per ounce.
- Since all investors can follow the same strategy, the London place would receive a large number of sell orders for gold while the NY market would receive a large number of purchase orders.
- The price in London will decrease while the price in NY will increase until both financial places offer the same price. In equilibrium, there cannot be arbitrage opportunities.

Def: The **law of one price** says that if an asset is traded simultaneously on two competitive markets, its price, in equilibrium, will be the same on both markets.

The no arbitrage condition also implies the additivity of values. Soit A and B two individual stocks and AB a portfolio composed of these two stocks. The additivity principle implies that:

$$V(AB) = V(A + B) = V(A) + V(B)$$

Indeed, suppose that $V(A) + V(B) < V(AB)$. An investor could sell (short) the portfolio AB , buy the individual stocks A and B and earn a positive profit without taking any risk. This will also be illustrated with numerical examples in the next section.

2 Financial mathematics: Refresher

2.1 Interest Rate

- Fundamental concept in finance: **interest rate**.
- The interest rate is the price of capital resulting from the supply (savings) and demand (investment) of economic agents.

There exists different methods to calculate the interests on an investment or a loan:

1. **Simple interests:** Interests are calculated on the initial capital and are paid at the end of each period (week, month, semester, year). Interests are not reinvested. Used to calculate the interests on bonds or short term financial operations (< 1 year).

Let K be the initial investment, r the interest rate and N the number of periods. The value of an investment is equal to:

- after 1 year: $K + Kr = K(1 + r)$
- after 2 years: $K + Kr + Kr = K(1 + 2r)$
- ...
- after N years: $K(1 + Nr)$

2. **Compounded interests:** Interests are calculated on the initial capital and accumulated interests at the end of every period. Interests are then reinvested every period.

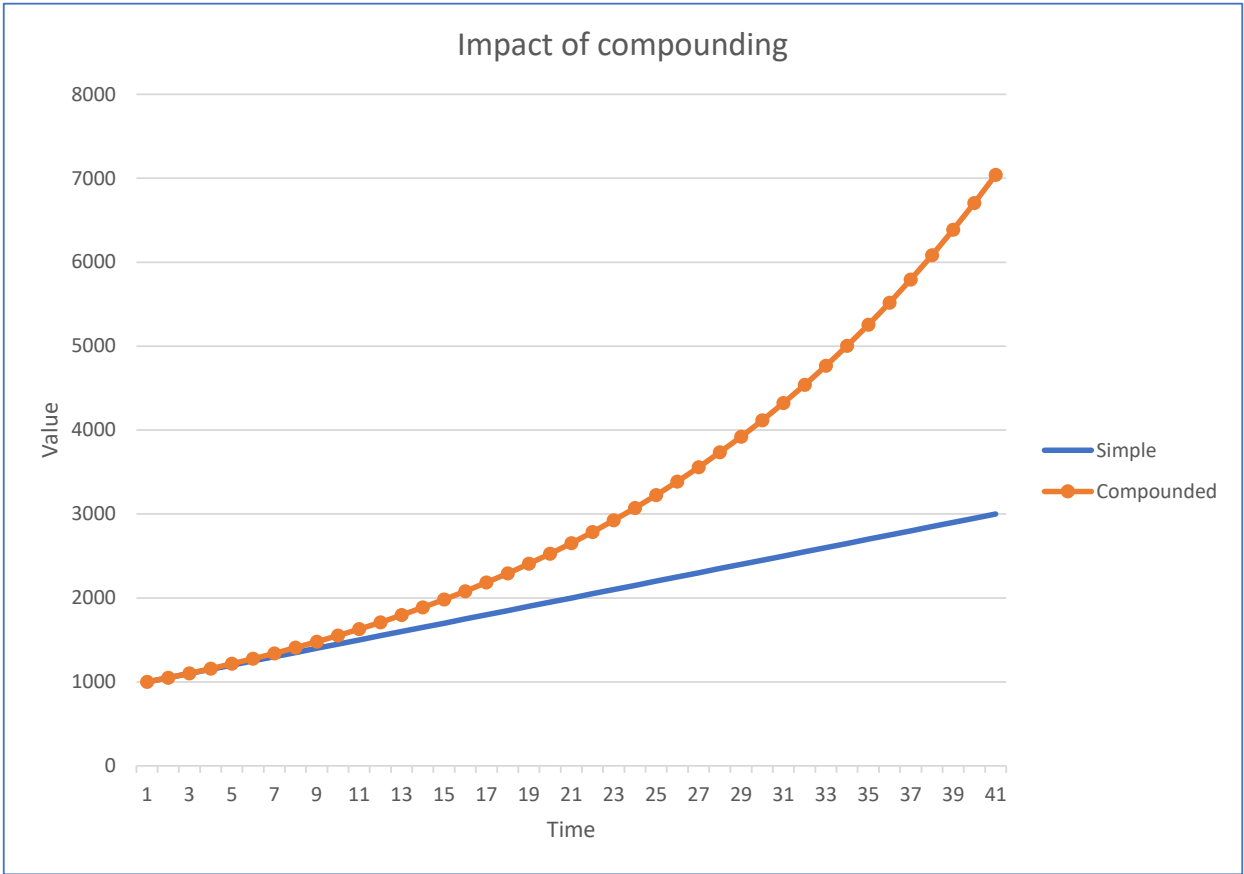
Let K be the initial investment, r the interest rate and N the number of periods. The value of an investment is equal to:

- after 1 year: $K(1 + r)$
- after 2 years: $K(1 + r)^2$
- ...
- after N years: $K(1 + r)^N$

Therefore, if the interests are calculated more than once during a year, the final wealth resulting from compounded interests is larger than the one from simple interests because of the interests reinvestment.

Example: Consider an investment of 1 000€ at the annual interest rate of 5%.

Years	Accumulated wealth Simple interests	Accumulated wealth Compounded interests
0	1 000	1 000
1	$1\,000(1.05) = 1\,050$	$1\,000(1.05)^1 = 1\,050$
2	$1\,000(1.10) = 1\,100$	$1\,000(1.05)^2 = 1\,103$
3	$1\,000(1.15) = 1\,150$	$1\,000(1.05)^3 = 1\,158$
4	$1\,000(1.20) = 1\,200$	$1\,000(1.05)^4 = 1\,216$
5	$1\,000(1.25) = 1\,250$	$1\,000(1.05)^5 = 1\,276$
...	...	...
10	$1\,000(1.50) = 1\,500$	$1\,000(1.05)^{10} = 1\,629$
20	$1\,000(2.00) = 2\,000$	$1\,000(1.05)^{20} = 2\,653$
30	$1\,000(2.50) = 2\,500$	$1\,000(1.05)^{30} = 4\,322$
40	$1\,000(3.00) = 3\,000$	$1\,000(1.05)^{40} = 7\,040$



Def: The **nominal** interest rate (r) is the contractual rate expressed on an annual basis and does not take into account the compounding of interests. This rate includes the expected inflation.

Def: p the number of compounding periods within a year, that is the number of times the interests are calculated during a year. Ex: $p = 1$ if the interests are annual; $p = 2$ if the interests are semi-annual; $p = 12$ if the interests are monthly.

Def: The **proportional** rate (r_p) is the interest rate corresponding to the period over which the interests are calculated. It is equal to the nominal rate (annual) divided by the number of times the interests are calculated over the year.

Def: The **effective** rate (r_e) is the interest rate which expresses the proportional rate on a yearly basis by taking into account the compounding of interests. This rate indicates the total amount of interests that will be earned after one year.

We can then write:

$$r_p = \frac{r}{p}$$

$$r_e = (1 + r_p)^p - 1 = \left(1 + \frac{r}{p}\right)^p - 1$$

One should note that the effective interest rate is higher than the nominal interest rate if $p > 1$. Indeed, if interests are compounded more than once a year, the investor receives interests on interests which generates a higher return than the nominal interest rate.

Example: Suppose an investment over a year where the interests are calculated on a semi-annual basis (2 times a year) at a semi-annual rate of 5%.

- Frequency of capitalization, p : 2.
- Proportionnal rate, r_p : 5%
- Nominal rate: r : $p \times r_p = 10\%$
- Effective rate, r_e : $(1 + r_p)^p - 1 = (1.05)^2 - 1 = 10.25\%$

2.2 Present Value

General rule: A stream of cash flows $\{F_0, F_1, F_2, \dots, F_N\}$ is defined over many periods. They must then be expressed on a same time basis in order to compare or aggregate the different cash flows.

In addition, we know that a future cash flow is worth less than a present cash flow. Why?

1. Preference for the present of agents (assumption)

- It is generally assumed that agents have a preference for the present.
- Having a preference for the present does not imply that an agent will always choose to consume everything today. It means that he is willing to postpone his consumption if he is correctly compensated (positive interest) for that sacrifice.
- It can be shown that an agent who has a preference for the present will exhibit a positive subjective interest rate which he can then compare with the market interest rate to decide whether or not to postpone his consumption.

2. Risk aversion (assumption)

- Future cash flows are more risky than present cash flows. A risk averse agent prefers 1€ with certainty than 1€ with uncertainty.
- Risk aversion does not imply that agents are not willing to take risk. Indeed, they are if they are correctly compensated for the risk that they are taking.

3. Inflation

Inflation reduces the value of future cash flows.

Def: The **Present Value (PV)** of a future cash flow F_t , discounted at rate r , over t periods is given by:

$$PV = \frac{F_t}{(1+r)^t}$$

The term $1/(1+r)^t$ is called the **discount factor** and r , the **discount rate** (nominal or proportional).

Examples:

- 1) The present value of a cash flow of 100 paid in 1 year with an annual interest rate $r = 6\%$

$$PV = \frac{F_1}{(1+r)^1} = \frac{100}{(1.06)^1} = 94.34$$

- 2) The present value of a cash flow of 500 paid in 2 years with an annual interest rate $r = 9\%$

$$PV = \frac{F_2}{(1+r)^2} = \frac{500}{(1.09)^2} = 420.84$$

3) The present value of a first cash flow of 100 paid in 1 year and a second cash flow of 200 paid in 2 years with an annual interest rate $r = 10\%$

$$PV = \frac{F_1}{(1+r)^1} + \frac{F_2}{(1+r)^2} = \frac{100}{(1.1)^1} + \frac{200}{(1.1)^2} = 256.2$$

Impact of time on present value.

Consider a financial asset paying 1 000€. The annual discount rate is 10%. The following table reports the present value as a function of the number of years to receive the payment.

Year	Present value
2	$1\,000/(1.1)^2 = 826.5$
10	$1\,000/(1.1)^{10} = 385.5$
25	$1\,000/(1.1)^{25} = 92.3$
50	$1\,000/(1.1)^{50} = 8.5$
100	$1\,000/(1.1)^{100} = 0.07$

Generally, the present value of a stream of cash flows $\{F_1, F_2, \dots, F_N, \}$ is equal to:

$$PV = \frac{F_1}{(1+r)^1} + \frac{F_2}{(1+r)^2} + \dots + \frac{F_N}{(1+r)^N} = \sum_{t=1}^N \frac{F_t}{(1+r)^t}$$

This represents the **additivity principle** of the present value.

Def: The **Net Present Value (NPV)** of a stream of cash flows $\{F_0, F_1, F_2, \dots, F_N, \}$, discounted at a rate r over N years is given by:

$$NPV = F_0 + \frac{F_1}{(1+r)^1} + \frac{F_2}{(1+r)^2} + \dots + \frac{F_N}{(1+r)^N} = \sum_{t=0}^N \frac{F_t}{(1+r)^t}$$

Remarks:

- We refer to the Net Present Value (NPV) when the initial investment ($F_0 < 0$) is included in the calculation.
- Future cash flows are usually positive (inflows) but some can be negative (outflows).

2.3 Some useful formulas

Discounting of a long stream of cash flows can sometimes be cumbersome. There exists useful mathematical formulas which, under certain assumptions, makes the calculations easier. The formulas derive from the properties of geometric series.

Def: A geometric series is a series of numbers with a constant ratio between successive elements. Each element of the series is equal to the previous element times a constant called the **common ratio**.

Ex: $\{2, 4, 8, 16, 32, 64, 128, 256, \dots\}$ is a geometric series of with a common ratio equal to 2.

Let F_n be the number in the n th position of the series of common ratio equal to q . We can then write $F_n = q F_{n-1}$. Similarly, $F_{n-1} = q F_{n-2}$, $F_{n-2} = q F_{n-3}$,

By substitution, we get $F_n = q^n$.

In the context of financial mathematics, we are mainly concerned by the sum of a given number of elements of a series.

1. Finite sum of the N first elements of a geometric series

$$q + q^2 + q^3 + \dots + q^N = \sum_{t=1}^N q^t = \frac{q(1 - q^N)}{(1 - q)}$$

Special case:

a) $q = 1/(1 + r)$

We have:

$$PV = \frac{1}{(1 + r)} + \frac{1}{(1 + r)^2} + \frac{1}{(1 + r)^3} + \dots + \frac{1}{(1 + r)^N}$$

$$PV = \sum_{t=1}^N \frac{1}{(1 + r)^t} = \frac{1}{r} \left(1 - \frac{1}{(1 + r)^N} \right) \quad \text{Constant annuity}$$

b) $q = (1 + g)/(1 + r)$, where g is the constant growth rate in the cash flow

We have:

$$PV = \frac{(1 + g)}{(1 + r)} + \frac{(1 + g)^2}{(1 + r)^2} + \frac{(1 + g)^3}{(1 + r)^3} + \dots + \frac{(1 + g)^N}{(1 + r)^N}$$

$$PV = \sum_{t=1}^N \frac{(1+g)^t}{(1+r)^t} = \frac{(1+g)}{r-g} \left(1 - \frac{(1+g)^N}{(1+r)^N} \right) \quad \text{Growing annuity}$$

Remarks on annuity formulas:

- These formulas are worth for a cash flow of 1 euro. For a flow equal to F , the formula is multiplied by F .
- The annuity formulas apply only for a constant cash flow, interest rate and growing rate.
- For the growing annuity, it is assumed that the growing rate applies starting today. In the event that the growth rate applies only starting next year, the formula must be divided by $(1+g)$.
- The growing annuity formula applies only for $r > g$.

Examples:

a) Calculate the PV of an annual cash flow of 200 euros over 20 years if $r = 8\%$:

We can use the classic approach:

$$PV = \frac{200}{(1,08)} + \frac{200}{(1,08)^2} + \dots + \frac{200}{(1,08)^{19}} + \frac{200}{(1,08)^{20}} = 1\,963.63$$

or the constant annuity approach,

$$PV = \frac{200}{0.08} \left(1 - \frac{1}{(1,08)^{20}} \right) = 1\,963.63$$

b) Calculate the PV of an annual cash flow of 500 euros over 10 years if $r = 6\%$ and $g = 2\%$ starting on the first payment:

We can use the classic approach:

$$PV = \frac{500(1,02)}{(1,06)} + \frac{500(1,02)^2}{(1,06)^2} + \dots + \frac{500(1,02)^9}{(1,06)^9} + \frac{500(1,02)^{10}}{(1,06)^{10}} = 4\,071.33$$

or the growing annuity approach:

$$PV = \frac{500(1.02)}{(0.06 - 0.02)} \left(1 - \frac{(1.02)^{10}}{(1.06)^{10}} \right) = 4\,071.33$$

2. Infinite sum of the elements of a geometric series

We apply the previous formulas with $N \rightarrow \infty$. We know that:

$$\lim_{N \rightarrow \infty} q^N = 0 \quad \text{if } q < 1$$

In the discounting problems, we have $q = 1/(1+r) < 1$. Hence,

a) $q = 1/(1+r)$:

$$\lim_{N \rightarrow \infty} \frac{1}{r} \left(1 - \frac{1}{(1+r)^N} \right) = \frac{1}{r} \quad \text{Constant perpetuity}$$

b) $q = (1+g)/(1+r)$, where g is the constant growth rate in the cash flow

$$\lim_{N \rightarrow \infty} \frac{(1+g)}{r-g} \left(1 - \frac{(1+g)^N}{(1+r)^N} \right) = \frac{(1+g)}{r-g} \quad \text{Growing perpetuity}$$

Note: The growing perpetuity formula requires that $r > g$. Otherwise, we need to take a different approach.

Examples:

a) Calculate the PV of an annual cash flow of 200 euros to infinity if $r = 7\%$:

$$PV = \frac{200}{0.07} = 2\,857.14$$

b) Calculate the PV of an annual cash flow of 100 euros to infinity if $r = 9\%$ and $g = 3\%$ starting on the first year:

$$PV = \frac{100(1.03)}{0.06} = 1\,716.67$$

Remarks:

- The PV of a stream of cash flows corresponds to the price an agent would be willing to pay in order to receive these cash flows.
- **The PV of a stream of cash flows to infinity is not equal to infinity.** Even if the total amount received by an agent is equal to infinity, the price for this stream of cash flows is finite. This is the result of discounting.

Present Value: Application 1 (In class)

A financial asset pays a cash flow of \$500 for the first four years and \$600 for the following four years. The cash flows are paid at the end of each year. The annual discount rate corresponding to the risk of the asset is 6%. Calculate the price of the asset.

Present Value: Application 2 (Home)

A financial asset pays a cash flow of \$200 for the first five years (end of year payments) and then \$100 per year to infinity. The annual discount rate corresponding to the risk of the asset is 5%. Calculate the price of the asset.

Question: What about if the cash flows are monthly or semi-annual?

Answer: Cash flows must be discounted at the proportional rate r_p .

Consider the previous example with an investment paying off 200 euros annually over 20 years. Instead of receiving 200 euros a year, the investor receives 100 euros every six months. The semi-annual interest rate is 4%.

The semi-annual cash flow is then 100 paid over 40 semesters and invested at a proportional rate of 4%:

Classic formula:

$$PV = \frac{100}{(1.04)} + \frac{100}{(1.04)^2} + \cdots + \frac{100}{(1.04)^{39}} + \frac{100}{(1.04)^{40}} = 1\,979.28$$

Annuity (semi-annual) formula:

$$PV = \frac{100}{0.04} \left(1 - \frac{1}{(1.04)^{40}} \right) = 1\,979.28$$

Remark: The PV of a steam of semi-annual cash flows is greater than the PV of annual cash flows. Indeed, semi-annual cash flows can be reinvested more often hence, generating a higher PV.

Conclusion: The PV of a stream of cash flows increases with the frequency of capitalization.

Example of an annuity with monthly payments: **bank loan**.

Consider a bank loan for 10 000 euros over two years at a nominal rate of 7%. Payments are made on a monthly basis at a proportional rate of $0.07/12 = 0.00583$.

i) Calculation of monthly payments, F :

Using the annuity formulas, we have:

$$\begin{aligned}PV &= \frac{F}{r} \left(1 - \frac{1}{(1+r)^n} \right) \\10\,000 &= \frac{F}{0.00583} \left(1 - \frac{1}{(1.00583)^{24}} \right) \\F &= 447.73\end{aligned}$$

ii) Calculation of the balance to be repaid at the beginning of the second year:

At the beginning of year 2, there are still 12 monthly payments of 447.73 being due. We can calculate the PV of these payments:

$$\begin{aligned}\text{Balance} &= \frac{447.73}{0.00583} \left(1 - \frac{1}{(1.00583)^{12}} \right) \\&= 5\,174.42\end{aligned}$$

We can get the same result using an amortization table:

Date	Initial Balance	Payment	Interest	Capital	Final balance
1/01	10 000	447.73	58.33	389.39	9 610.61
1/02	9 610.61	447.73	56.06	391.66	9 218.94
1/03	9 218.94	447.73	53.78	393.95	8 825.00
1/04	8 825.00	447.73	51.48	396.25	8 428.75
1/05	8 428.75	447.73	49.17	398.56	8 030.19
1/06	8 030.19	447.73	46.84	400.88	7 629.31
1/07	7 629.31	447.73	44.50	403.22	7 226.09
1/08	7 226.09	447.73	42.15	405.57	6 820.51
1/09	6 820.51	447.73	39.79	407.94	6 412.57
1/10	6 412.57	447.73	37.41	410.32	6 002.25
1/11	6 002.25	447.73	35.01	412.71	5 589.54
1/12	5 589.54	447.73	32.61	415.12	5 174.42

Important remark: A risky cash flow must be discounted at a rate corresponding to its risk. Hence, a risk free cash flow must be discounted at the risk free rate while a risky cash flow must be discounted at the risk free rate plus a risk premium (see the section on CAPM).

2.4 Applications to financial assets

- Stocks

Stockholders are entitled to receive future dividends and a future price at the time they sell their stocks.

The market price of a stock is then equal to the present value of future expected dividends and a future expected price. Let the investment horizon be equal to H , we can write:

$$P = \frac{E(\tilde{D}_1)}{(1+r)} + \frac{E(\tilde{D}_2)}{(1+r)^2} + \frac{E(\tilde{D}_3)}{(1+r)^3} + \cdots + \frac{E(\tilde{D}_H)}{(1+r)^H} + \frac{E(\tilde{P}_H)}{(1+r)^H}$$

or alternatively,

$$P = \sum_{t=1}^H \frac{E(\tilde{D}_t)}{(1+r)^t} + \frac{E(\tilde{P}_H)}{(1+r)^H}$$

We can also write that

$$E(\tilde{P}_H) = \frac{E(\tilde{D}_{H+1})}{(1+r)} + \frac{E(\tilde{D}_{H+2})}{(1+r)^2} + \frac{E(\tilde{D}_{H+3})}{(1+r)^3} + \cdots$$

Therefore, replacing $E(\tilde{P}_H)$ in the stock price equation, we get,

$$P = \sum_{t=1}^{\infty} \frac{E(\tilde{D}_t)}{(1+r)^t}$$

Thus, the price of a stock is equal to the present value of an infinite stream of future expected dividends.

If we want to calculate this price accurately, we need to give a structure to the future expected dividends. This is the objective of the **Gordon** model which assumes that dividends increase at a constant annual growth rate equal to g . Thus, we can write,

$$\begin{aligned} - E(\tilde{D}_1) &= D_0(1+g) \\ - E(\tilde{D}_2) &= E(\tilde{D}_1)(1+g) = D_0(1+g)^2 \\ - &\vdots \\ - E(\tilde{D}_N) &= E(\tilde{D}_{N-1})(1+g) = D_0(1+g)^N \end{aligned}$$

where D_0 is the last dividend per share paid by the firm. Replacing in the stock price equation,

$$P = D_0 \left[\frac{(1+g)}{(1+r)} + \frac{(1+g)^2}{(1+r)^2} + \frac{(1+g)^3}{(1+r)^3} + \dots + \frac{(1+g)^N}{(1+r)^N} \right] \quad N \rightarrow \infty$$

This formula corresponds to a geometric series of common factor $(1+g)/(1+r)$. This solves to:

$$P = D_0 \frac{(1+g)}{(r-g)} = \frac{E(\tilde{D}_1)}{(r-g)}$$

Exemple: Consider a stock which recently paid a dividend per share $D_0 = 5$ which expects to grow at annual rate of 4% to infinity. Given the risk of this stock, the required expected rate of return by investors is equal to 10%. Therefore, we can write,

$$P = D_0 \frac{(1+g)}{(r-g)} = \frac{5(1.04)}{(0.10 - 0.04)} = 86.67$$

Question: How can this be used by investors or equity analysts on the market?

Present Value: Application 3 - Stock Pricing (In class)

The stock of the firm “La Vie Est Belle” recently paid a dividend of 3€ . Given that the firm faces significant growth in the coming years, the dividend is expected to grow at an annual rate of 5% over the next four years. At then end of this period (end of year 4), you expect the stock to sell at about 50€. The required rate of return by investors is 9%. Calculate the market price of that stock.

- Bonds

As seen earlier, bonds pay interests (coupons) and a principal. Thus, the market price of a bond is equal to the present value of coupons and the principal (or nominal value). We have seen that the stream of cash flows associated to a bond is given by:

$$\{P_0, C_1, C_2, C_3, \dots, C_N, NV_N\}$$

We can then write that:

$$P = \sum_{t=1}^N \frac{C_t}{(1+r)^t} + \frac{NV}{(1+r)^N}$$

Assuming *in fine* fixed interest rate bonds, the coupons are fixed over N years, we can write this formula as:

$$P = \frac{C}{r} \left[1 - \frac{1}{(1+r)^N} \right] + \frac{NV}{(1+r)^N}$$

where r is equal to the discount rate which reflects the risk of the bond, commonly called the yield to maturity (internal rate of return).

Example: Suppose a fixed interest bond with a nominal value of \$5 000 and offering an annual coupon (interest) rate of 10%. The bond comes to maturity in 4 years and the required rate of return by investors on the market (yield to maturity) is 8%.

The annual coupon on this bond is: $5\,000 \times 10\% = 500$. The price of this bond is then equal to:

$$P = \sum_{t=1}^N \frac{C}{(1+r)^t} + \frac{NV}{(1+r)^N} = \sum_{t=1}^4 \frac{500}{(1.08)^t} + \frac{5\,000}{(1.08)^4} = 5\,331.21$$

or alternatively,

$$P = \frac{500}{0.08} \left[1 - \frac{1}{(1.08)^4} \right] + \frac{5\,000}{(1.08)^4} = 5\,331.21$$

Present Value: Application 4 - Bond Pricing (In class)

A bond with a nominal value of \$10 000 offers an annual coupon (nominal interest) rate of 5%. The bonds comes to maturity in 15 years and offers a yield to maturity (discount rate) of 4%. Calculate the price of this bond. What about if the yield to maturity was 5%? And 6%? What do you conclude?

				Bond Ratings		
Moody's		Standard & Poors		Fitch		
Long term	Short term	Long term	Short term	Long term	Short term	Description
Aaa	P-1	AAA	A-1+	AAA	F1+	Prime
Aa1		AA+		AA+		High Grade
Aa2		AA		AA		
Aa3		AA-		AA-		
A1		P-2	A+	A-1	A+	F1+
A2	A		A			
A3	A-		A-2	A-	F2	Lower Medium Grade
Baa1	BBB+	BBB+		F3		
Baa2	P-3	BBB			BBB	
Baa3		BBB-	A-3	BBB-		
Ba1	Not Prime	BB+	B	BB+	B	Non-Investment Grade Speculative
Ba2		BB		BB		
Ba3		BB-		BB-		Highly Speculative
B1		B+		B+		
B2		B		B		
B3		B-		B-		
Caa1		C	CCC+	C	CCC+	Substantial Risks
Caa2			CCC		CCC	
Caa3			CCC-		CCC-	
Ca			CC		CC	Extremely Speculative
			C		C	Default Imminent
C		D	RD	D	DDD	In Default
--			SD		DD	
--			D		D	

2.5 Future Value

Future value is the inverse operation to present value.

Def: The **future value** (FV) or **final value** of a cash flow F , compounded at the rate r over T periods, is given by the expression:

$$FV = F (1 + r)^T$$

The term $(1 + r)^T$ is said to be the compound factor.

Special cases.

a) Future value at time T of an investment F_s at period s :

$$FV = F_s (1 + r)^{(T-s)}$$

Example: Calculate the FV in 2025 of an investment of 3 000 euros invested in 2015 at an annual rate of 10%.

$$FV = 3\,000 (1.1)^{10} = 7\,781.23$$

b) Future value at time T of a constant annuity, F , with the first payment at $s + 1$:

$$FV = \frac{F}{r} [(1 + r)^{(T-s)} - 1]$$

This formula can be decomposed in three elements:

$$\frac{F}{r} \left(1 - \frac{1}{(1 + r)^{T-s}} \right) \quad \text{PV at time } s \text{ of a stream of cash flows starting at the end of year } s$$

$$\frac{1}{(1 + r)^s} \quad \text{PV at time 0 of the a above PV}$$

$$(1 + r)^T \quad \text{Future value at time } T \text{ of the a above PV}$$

Multiplying the three terms, we get:

$$\frac{F}{r} \left(1 - \frac{1}{(1 + r)^{T-s}} \right) \times \frac{1}{(1 + r)^s} \times (1 + r)^T = \frac{F}{r} [(1 + r)^{(T-s)} - 1]$$

Example: Calculate the future value at the end of year 2050 of an annual investment of 1 000 euros starting at the end of year 2010 if $r = 8\%$.

$$FV = \frac{1\,000}{0,08} [(1,08)^{40} - 1] = 259\,057$$

2.6 Arbitrage and the Law of One Price

In the above section, we have argued that the price of any asset is equal to the present value of the cash flows generated by that asset. In that context, the **Law of One Price** implies that any two assets generating the same cash flows and having the same risk must have the same price. Otherwise, there will be arbitrage opportunities whereby an investor could earn a positive profit without taking risk. Consider the following examples (Berk & DeMarzo):

Example: A risk free security pays 1 000 in one year. The risk-free interest rate is 5%.

Questions

- i) What is the no-arbitrage price of that security ?
- ii) If the security is trading at 940, is there an arbitrage opportunity?

Answers

i) The no-arbitrage price of a security is equal to the present value of the cash flows paid to the investor.

$$\text{Price} = \frac{1\,000}{1.05} = 952.38$$

ii) Given that the security trades at 940 (undervalued), there is an arbitrage opportunity.

Consider the following strategy:

- Borrow from the bank an amount equal to the no arbitrage price of the bond: 952.38
- Buy the bond at the actual market price: 940

The following table summarizes the cash flow structure:

Strategy	F_0	F_1
Bank Loan	952.38	-1 000
Buy the bond	-940	1 000
Net cash flows	12.38	0

Conclusion: Any investor can follow this strategy. The increased demand for the bond will raise its price and this will continue until it reaches 952.38 and there is no more arbitrage opportunity.

Present Value: Application 5(a) - Arbitrage strategy (In class)

A risk free security pays 1 000 in one year. The risk-free interest rate is 5%. Now, assume that the bond is trading at 960 (overvalued). Construct an arbitrage strategy.

Present Value: Application 5(b) - Arbitrage strategy (Home)

A risk-free security pays 100 today and 100 in one year. The risk-free interest rate is 5%.

- i) What is the no-arbitrage price of that security (before the first 100 is paid)?**
- ii) If the security is trading at 198, is there an arbitrage opportunity?**

2.7 Beginning and end of year payments

Consider a constant annuity over N periods.

Case 1: The cash flows are paid at the end of each period. The first cash flow is due one period from now:

$$PV = \frac{F}{(1+r)} + \frac{F}{(1+r)^2} + \frac{F}{(1+r)^3} + \dots + \frac{F}{(1+r)^N}$$

Cas 2: The cash flows are paid at the beginning of each period. The first cash flow is due now:

$$PV = F + \frac{F}{(1+r)} + \frac{F}{(1+r)^2} + \dots + \frac{F}{(1+r)^{N-1}}$$

As one can notice, the PV (beginning of period) is equal to the PV (end of period) times $(1+r)$. We can then go from one formula to another by multiplying or dividing by a factor equal to $(1+r)$.

An Old Joke

There is an old joke that many finance professors enjoy telling their students. It goes like this:

A finance professor and a student are walking down a street. The student notices a \$100 bill lying on the pavement and leans down to pick it up. The finance professor immediately intervenes and says, "Don't bother; there is no free lunch. If that were a real \$100 bill lying there, somebody would already have picked it up!"

This joke invariably generates much laughter because it makes fun of the principle of no arbitrage in competitive

markets. But once the laughter dies down, the professor then asks whether anyone has ever *actually* found a real \$100 bill lying on the pavement. The ensuing silence is the real lesson behind the joke.

This joke sums up the point of focusing on markets in which no arbitrage opportunities exist. Free \$100 bills lying on the pavement, like arbitrage opportunities, are extremely rare for two reasons: (1) Because \$100 is a large amount of money, people are especially careful not to lose it, and (2) in the rare event when someone does inadvertently drop \$100, the likelihood of your finding it before someone else does is extremely small.

Arbitrage in Markets

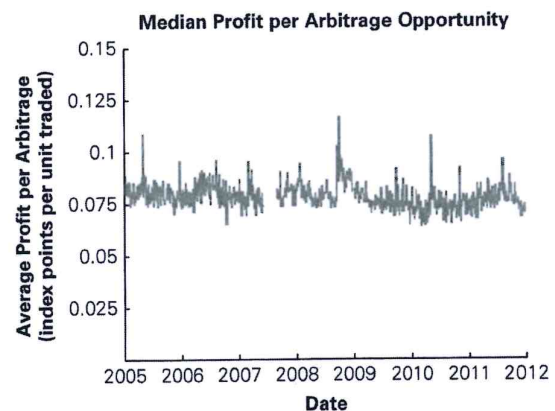
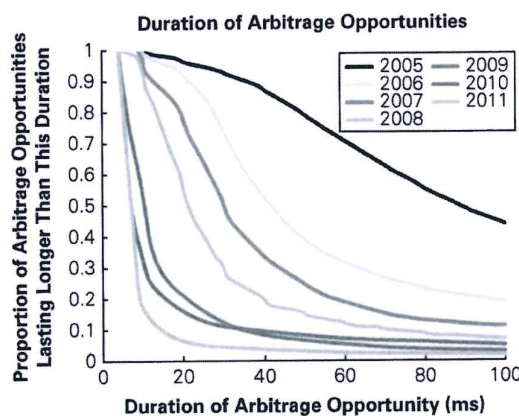
Value additivity is the principle behind a type of trading activity known as stock index arbitrage. Common stock indices (such as the Dow Jones Industrial Average and the Standard and Poor's 500 (S&P 500)) represent portfolios of individual stocks. It is possible to trade the individual stocks that comprise an index on the New York Stock Exchange and NASDAQ. It is also possible to trade the entire index (as a single security) on the futures exchanges in Chicago, or as an exchange-traded fund (ETF) on the NYSE. When the price of the index security is below the total price of the individual stocks, traders buy the index and sell the stocks to capture the price difference. Similarly, when the price of the index security is above the total price of the individual stocks, traders sell the index and buy the individual stocks. It is not uncommon for 20% to 30% of the daily volume of trade on the NYSE to be due to index arbitrage activity via program trading.*

The traders that engage in stock index arbitrage automate the process by tracking prices and submitting (or cancelling) orders electronically. Over the years the competition to take advantage of these opportunities has caused traders to go to extraordinary lengths to reduce order execution time. One limiting factor is the time it takes to send an order from one exchange to another. For example, in 2010 Spread Networks paid \$300 million for a new fiber optic line that reduced the communication time between New York and Chicago from 16 milliseconds to 13 milliseconds. Three milliseconds might not sound like a lot (it takes 400 milliseconds to blink), but it meant that Spread would be able to exploit mispricings that occurred between the NYSE and the Chicago futures exchange before any of its competitors, at least until one of its competitors constructed a faster line.

The evolution of how traders took advantage of these short-lived arbitrage opportunities provides a nice illustration of how competitive market forces act to remove profit-making opportunities. In a recent study, Professors Eric Budish, Peter Crampton, and John Shim** focused on the evolution of one particular arbitrage opportunity that resulted from differences in the price of the S&P 500 Futures Contract on the Chicago Mercantile Exchange and the price of the SPDR S&P 500 ETF traded on the New York Stock Exchange.

The left figure shows how the duration of arbitrage opportunities changed between 2005 and 2011. Each line shows, for the indicated year, the fraction of arbitrage opportunities that lasted longer than the amount of time indicated on the horizontal axis. So, for example, in 2005 about half of the arbitrage opportunities that existed lasted more than 100 milliseconds. By 2008, this number had dropped to 20 milliseconds, and by 2011, the number was under 10 milliseconds. Note also that in 2005 almost all opportunities lasted at least 20 milliseconds, but by 2011 the number of opportunities that lasted this long was less than 10% and hardly any persisted for more than 100 milliseconds.

What happened to the profits from exploiting these mispricings? You might have expected that the effect of this competition would be to decrease profits, but as the right figure shows, profits per opportunity remained relatively constant. Furthermore, the number of opportunities did not systematically decline over this period, implying that the aggregate profits from exploiting arbitrage opportunities did not diminish. In that sense, the competition between arbitrageurs has not reduced the magnitude or frequency of price deviations across these markets, but instead has reduced the amount of time that these deviations can persist.



*See www.barrons.com/public/page/9_0210-nysepgtd.html

**"The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response," *Quarterly Journal of Economics* (2015): 1547-1621.

3 Capital Budgeting (Investing) under certainty

A generally accepted principle is that a firm aims at maximizing the value of its equity. Thus, an investment project should be undertaken if it creates value for equityholders.

Question: Should a firm undertake an investment project?

Two commonly used criteria can be used to answer this question:

1. Net Present value (NPV)
2. Internal Rate of Return (IRR)

3.1 Net Present Value (NPV)

Def: Defining F_0 as the initial investment of a project, the NPV is equal to:

$$NPV = -F_0 + \sum_{t=1}^N \frac{F_t}{(1+r)^t}$$

Decision rule:

- If $NPV > 0$: undertake the project,
- Otherwise: reject the project.

There are a number of essential rules to follow when conducting a capital budgeting analysis.

1. Consider only cash flows.

- Inflows
- Outflows

We are not interested in accounting flows per se but only in cash flows.

2. Consider only the marginal cash flows associated to a specific project.

The evaluation is done relative to a base case scenario which could already have cash flows or not. Only the variation in cash flows must be accounted for.

	Before project	After project	Δ Cash flows
Total sales	100 000	150 000	50 000
Total cost	80 000	110 000	30 000
Cash flows	20 000	40 000	20 000

The cash flows associated to the project are equal to 20 000 since they correspond to the additional cash flows brought in by the new investment.

3. Cash flows are independent of the financing of the project.

The use of debt or equity financing does not impact on the firm's cash flows. Thus, the firm's cash flows are independent of the interests, dividends and debt payments. The financing decisions will be captured by the cost of capital (see §9).

4. Do not include sunk costs.

Costs that have already been incurred but which have no impact on the future cash flows should not be considered in the analysis. For example, market studies or other expenses which have already been paid for before the beginning of the project are lost and should not be considered for the analysis. They are accounting flows but not cash flows for the project.

5. Include the secondary or indirect effects of the investment.

A new investment may have an impact (positive or negative) on the existing cash flows of the firm. This should be included in the analysis.

6. All cash flows are measured in market value.

3.2 Cash flows

The most important element in investment analysis is the measurement of cash flows. They can be decomposed in four general categories.

- Capital expenditures (CAPEX)
- Operational cash flows
- Variation in working capital requirement (non-cash working capital)
- Sales of assets

1. Capital expenditures (CAPEX)

- Most of the investments typically arise at $t = 0$. Yet, a project may include maintenance investments over the life of the project.
- They include tangible (machines, land, buildings, tools, ...) and intangible (patents, trademarks) assets.
- They are estimated on a market value or opportunity cost basis.

2. Operational cash flows

	Sales	
-	Cost of goods sold (COGS)	
-	Expenses	
	Earnings before interests, taxes, depreciation & amortization (EBITDA)	
-	Depreciation & amortization (D&A)	
	Earnings before interests & taxes (EBIT)	
-	Taxes on EBIT	
	EBIT after tax (Net operating profit after tax (NOPAT))	
+	Depreciation & amortization (D&A)	
	Operational cash flows	

Note: **Depreciation** refers to the decrease in the value of tangible assets while **amortization** is used for intangible assets.

Mathematically, we can write:

$$CF_{op} = EBIT(1 - \tau) + D\&A$$

Given that $EBIT = EBITDA - D\&A$, we can also write that,

$$CF_{op} = EBITDA(1 - \tau) + \tau D\&A$$

where τ is the corporate tax rate.

Both approaches are equivalent. Yet, the EBITDA approach stresses the fact that although $D\&A$ is not a cash flow (it is an accounting flow), the tax savings on $D\&A$ is a cash flow.

There are different methods to account for the depreciation of assets.

- Linear
- Declining

For simplicity, we will use only linear depreciation.

$$D\&A = \frac{\text{Value of asset}}{\text{Number of years}}$$

Example: You buy an asset at a price of 100 000€ and its expected life is 5 years. The annual depreciation that will be used for the investment analysis is 20 000€.

The linear depreciation rate is equal to $1/N = 0.2$

3. Working capital requirement (WCR)

Any investment which generates cash flows will have an impact on the working capital requirement (WCR) also called the non-cash working capital.

The working capital requirement (WCR) for operational purposes is part of the working capital (WC).

$$WC = (\text{Inventories} + \text{Receivables} - \text{Payables}) + (\text{Cash \& eq.} - \text{Short term financial debt})$$

$$WCR = \text{Inventories} + \text{Receivables} - \text{Payables} = I + R - P$$

Two important rules:

- Consider only the variations in WCR from one year to another.
- Recover the WCR at the end of the project (bring back to its original level)

The WCR includes short term assets and liabilities. They are accounted as follows:

↑ short term asset $\implies$ marginal outflow (-)

↓ short term asset $\implies$ marginal inflow (+)

↑ short term liability $\implies$ marginal inflow (+)

↓ short term liability $\implies$ marginal outflow (-)

Example:

t	WCR	ΔWCR
0	0	
1	15	15
2	20	5
3	23	3
4	18	-5
5	22	4

Thus, the present value of the ΔWCR is equal to:

$$PV = -\frac{15}{(1+r)} - \frac{5}{(1+r)^2} - \frac{3}{(1+r)^3} + \frac{5}{(1+r)^4} - \frac{4}{(1+r)^5} + \frac{22}{(1+r)^5}$$

Remarks:

- An increase in WCR has a negative impact on cash flows while a decrease has a positive impact.
- $\sum_{t=1}^N \Delta WCR = 0$. Yet, the present value is not equal to zero.

4. Sales of assets

Although a firm can sell assets over the entire life of a project, most of these sales will be at the very end.

- It is assumed that the firm will sell its remaining assets at the market price.
- Capital gains resulting from the sales of assets are taxed at the corporate tax rate.
- Capital gain (CG) = Sales price - net book value (NBV)
- Net book value = Purchase price - accumulated D&A

5. Cash flows to the firm

Taking into account all these elements, we can write:

$$\begin{array}{rcl} & - & \text{CAPEX} \\ & + & \text{EBITDA} \times (1 - \tau) \\ & + & \tau \times \text{D\&A} \\ & - & \uparrow I \text{ } (+ \downarrow I) \\ & - & \uparrow R \text{ } (+ \downarrow R) \\ & + & \uparrow P \text{ } (- \downarrow P) \\ & + & \text{Sales of assets (after tax on CG)} \\ \hline & & \text{Free cash flows} \end{array}$$

In a more compact form, we can write:

$$FCF = -CAPEX + EBITDA(1 - \tau) + \tau D\&A - \Delta WCR + \text{Sales assets (after tax)}$$

or alternatively,

$$FCF = -CAPEX + EBIT(1 - \tau) + D\&A - \Delta WCR + \text{Sales assets (after tax)}$$

Note: The Δ WCR includes the recovery of the WCR at the end of the project.

Capital Budgeting: Application (In class)

- The wine company Les Grands Vins Martel is considering the possibility to replace its old machinery with more productive equipment. There are presently two types of machine available on the market.

- Machine A:

- Purchase price (Capex): 750 000€
- Linear depreciation
- Increase in annual sales and production costs:

Year	Sales	Costs
1	300 000	50 000
2	300 000	50 000
3	300 000	50 000
4	300 000	50 000
5	300 000	50 000

- WCR: 5% of sales
- Estimated resale value (market) after 5 years: 100 000€

- Machine B:

- Purchase price (Capex): 1 000 000€
- Linear depreciation
- Increase in annual sales and production costs:

Year	Sales	Costs
1	400 000	50 000
2	400 000	50 000
3	400 000	50 000
4	400 000	50 000
5	400 000	50 000

- WCR: 5% of sales
- Estimated resale value (market) after 5 years: 150 000€

- Corporate tax rate: 30%.
- Cost of capital (discount rate): 10%

Compare these two alternatives on the basis of the NVP criteria.

3.3 Investment decisions with different time horizons

Many times, there exists different technologies to replace an existing one and these technologies may have different expected life.

Question: How do we compare investment projects with different time horizons?

Answer: There exists two equivalent approaches.

1. Equivalent annuity
2. Smallest common denominator of expected life

Let's illustrate these two methods with the following example.

- Technology X
 - Capex: - 5 000
 - Annual cash flows: 2 100
 - $N = 4$
- Technology Y
 - Capex: - 2 000
 - Annual cash flows: 1 500
 - $N = 3$
- Discount rate: 5%

Let's calculate the NPV for both projects.

$$\text{NPV}(X) = -5\,000 + \frac{2\,100}{0.05} \left[1 - \frac{1}{(1.05)^4} \right] = 2\,446.5$$

$$\text{NPV}(Y) = -2\,000 + \frac{1\,500}{0.05} \left[1 - \frac{1}{(1.05)^3} \right] = 2\,084.9$$

Conclusion: $X \succ Y$.

Problem: these two projects have different time horizons and it is possible that the choice of X over Y is driven by this factor and not by the underlying cash flows. To control for this problem, we can express these two projects on an equivalent time basis. This approach assumes that the projects can be renewed identically.

1. Equivalent annuity

The equivalent annuity approach is looking for the constant annuity that would generate the same NPV as the cash flows of the project. The Equivalent Annuity (EA) is the solution to:

$$NPV = \frac{EA}{r} \left[1 - \frac{1}{(1+r)^N} \right]$$

If we apply this to the previous example, we would get:

$$2\,446.5 = \frac{EA(X)}{0.05} \left[1 - \frac{1}{(1.05)^4} \right] \implies EA(X) = 689.94$$

$$2\,084.9 = \frac{EA(Y)}{0.05} \left[1 - \frac{1}{(1.05)^3} \right] \implies EA(Y) = 765.58$$

The interpretation of these results is quite simple.

- Project X

The sequence of cash flows (-5 000, 2 100, 2 100, 2 100, 2 100) has the same NPV as the sequence of cash flows (0, 689.94, 689.94, 689.94, 689.94). They both have the same NPV = 2 446.5

- Project Y

The sequence of cash flows (-2 000, 1 500, 1 500, 1 500) has the same NPV as the sequence of cash flows (0, 765.58, 765.58, 765.58). They both have the same NPV = 2 084.9

- Given that $EA(Y) > EA(X)$ every year to infinity, one can conclude that $Y \succ X$. This can be seen from the following table.

Time	CF(X)	EA(X)	CF(Y)	EA(Y)
0	-5 000		- 2 000	
1	2 100	689.94	1 500	765.58
2	2 100	689.94	1 500	765.58
3	2 100	689.94	-2 000 + 1 500	765.58
4	-5 000 + 2 100	689.94	1 500	765.58
5	2 100	689.94	1 500	765.58
6	2 100	689.94	-2 000 + 1 500	765.58
7	2 100	689.94	1 500	765.58
8	-5 000 + 2 100	689.94	1 500	765.58
⋮	⋮	⋮	⋮	⋮

2. Smallest common denominator of expected life

Under this approach, we assume that the two projects can be expressed on the same time finite time horizon. In the above example, the smallest common denominator is 12, that is the firm does Project X three times and Project Y four times.

We know that $NPV(X) = 2\,446.5$ and $NPV(Y) = 2\,084.9$ if it is done once. If we repeat these projects, we get:

- Project X is done at year 0, 4 and 8
- Project Y is done at year 0, 3, 6 and 9

$$NPV(X) = 2\,446.5 + \frac{2\,446.5}{(1.05)^4} + \frac{2\,446.5}{(1.05)^8} = 6\,115.12$$

$$NPV(Y) = 2\,084.9 + \frac{2\,084.9}{(1.05)^3} + \frac{2\,084.9}{(1.05)^6} + \frac{2\,084.9}{(1.05)^9} = 6\,785.55$$

Conclusion: $Y \succ X$.

3.4 Internal Rate of Return (IRR)

Def: The **IRR** of a project is the discount rate which sets the NPV to zero:

$$\text{IRR is } r \text{ such that: } NPV = -F_0 + \sum_{t=1}^N \frac{F_t}{(1+r)^t} = 0$$

Decision Rule:

- If $IRR > \text{cost of capital (WACC)}$: accept the project
- Otherwise: reject the project

3.4.1 How to find the IRR

Scientific calculators (or Excel) typically have a program to calculate the IRR. However, one should be careful as the program is usually specified for a given cash flow structure, namely a structure with an initial outflow (-) and subsequent annual inflows (+). If the cash flow structure differs from the previous one, the program may provide the wrong IRR or may not be able to calculate it accurately.

An alternative method to calculate the IRR is by using linear interpolation.

Method:

- Select any interest rate (r_1) and calculate the NPV of the sequence of cash flows (NPV_1)
- If the (NPV_1) is positive, this means that r_1 is too low since you are trying to arrive at a zero NPV. Select a higher interest rate ($r_2 > r_1$) such that the new NPV is negative (NPV_2).

- If the (NPV_1) is negative, this means that r_1 is too high. Select a lower interest rate ($r_2 < r_1$) such that the new NPV is positive (NPV_2).
- Using the fact that you have a positive and negative NPV, use linear interpolation (Thalès) to estimate the IRR.

Solution

$$\frac{NPV_1 - 0}{NPV_1 - NPV_2} = \frac{IRR - r_1}{r_2 - r_1} \quad \Rightarrow \quad IRR = r_1 + \frac{NPV_1(r_2 - r_1)}{NPV_1 - NPV_2}$$

Example: Consider the following sequence of cash flows: (-1 000, 400, 525, 345). The IRR is calculated as follows:

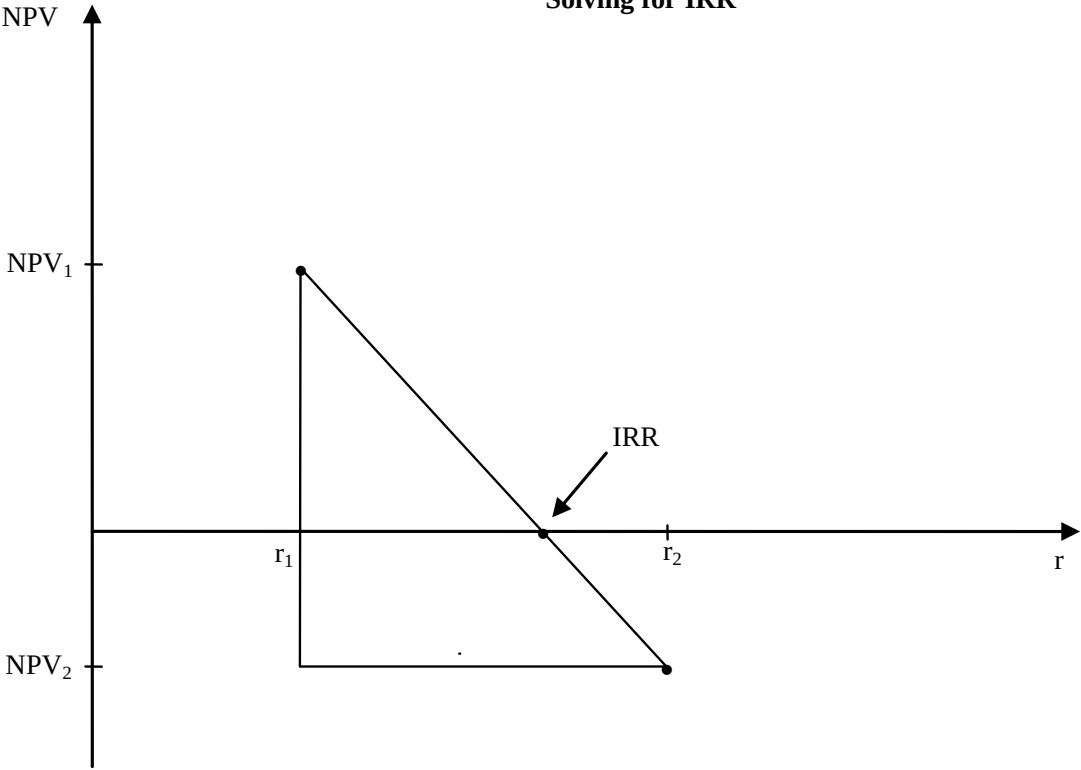
Step 1: Select a first interest rate and calculate the NPV: $r_1 = 0.1 \quad \Rightarrow \quad NPV_1 = 56.72$

Step 2: Select $r_2 > r_1$ and calculate the new NPV: $r_2 = 0.15 \quad \Rightarrow \quad NPV_2 = -28.36$

Step 3: Calculate the IRR

$$\frac{56.72}{56.72 + 28.36} = \frac{IRR - 0.1}{0.15 - 0.1} \quad \Rightarrow \quad IRR \sim 0.133$$

Solving for IRR



3.5 NPV vs. IRR

The NPV has many advantages:

1. reflects the discounting principle,
2. takes into account all the cash flows of a project,
3. uses the cost of capital for discounting,
4. measures the increase in a firm's value resulting from the investment,
5. respect the additivity principle.

There exist a necessary but not sufficient condition for the IRR criteria to give the same ranking as the NPV criteria: the NPV must be a monotone decreasing function of the interest rate.

Def: A function is **monotone decreasing** if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. If a function is monotone, the sign of its derivative $f'(x)$ is always the same and different than zero.

The IRR presents a few weaknesses and limitations in its use.

a) Mutually exclusive projects

When there are mutually exclusive projects, the IRR can generate a different ranking than the NPV. Example:

Cash Flow	A	B
F_1	-1 000	-1 000
F_2	500	100
F_3	500	100
F_4	500	1 450

i) NPV ($r = 10\%$)

$$NPV(A) = -1\,000 + \frac{500}{(1.1)^1} + \frac{500}{(1.1)^2} + \frac{500}{(1.1)^3} = 243$$

$$NPV(B) = -1\,000 + \frac{100}{(1.1)^1} + \frac{100}{(1.1)^2} + \frac{1\,450}{(1.1)^3} = 263$$

Conclusion: B is preferred to A .

ii) IRR

$$-1\,000 + \frac{500}{(1+r)^1} + \frac{500}{(1+r)^2} + \frac{500}{(1+r)^3} = 0 \Rightarrow IRR(A) = 23.4\%$$

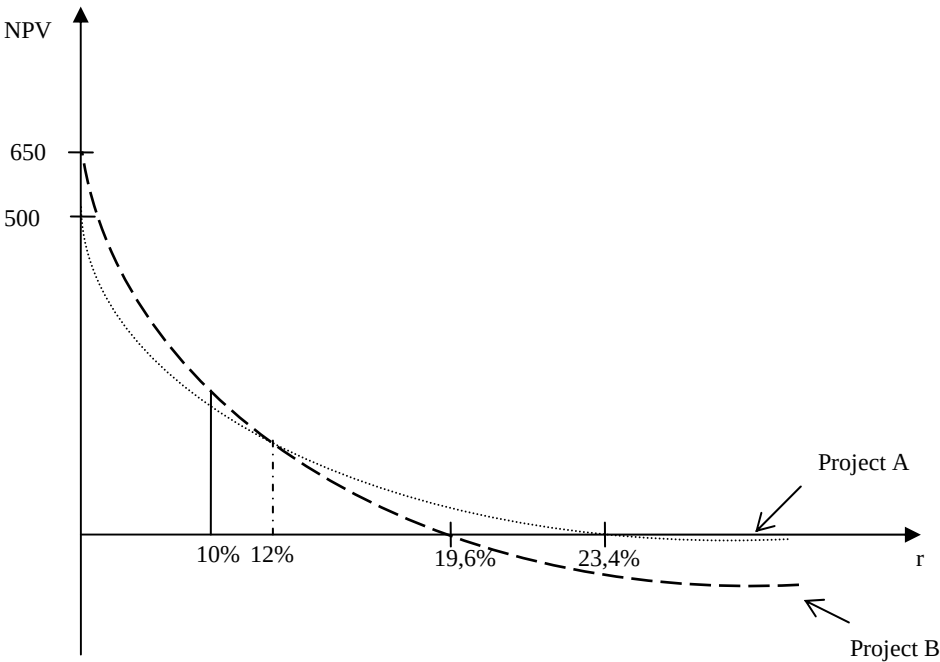
$$-1\,000 + \frac{100}{(1+r)^1} + \frac{100}{(1+r)^2} + \frac{1\,450}{(1+r)^3} = 0 \Rightarrow IRR(B) = 19.6\%$$

Conclusion: A is preferred to B .

The graph below illustrates the conflict between NPV and IRR. At cost of capital of 10%, the NPV(B) is greater than the NPV(A). On the basis of the NPV, A is preferred to B only if $r > 12\%$.

Conclusion: Since B creates more value at $r = 10\%$, it must be selected.

NPV vs.IRR



b) **Cannot make the difference between borrowing and lending**

Example:

Project	F_0	F_1	NPV (r=10%)	IRR
A (Lending)	- 1 000	1 500	364	50%
B (Borrowing)	1 000	-1 500	-364	50%

Based on the NPV, lending is preferred to borrowing money while these two operations are equivalent for the IRR. One should note that the NPV of B is a monotone increasing function of r .

c) **Many or no IRR**

General rule:

- If $F_0 < 0$ and $F_t > 0 \ \forall t$, there exist only one solution (single IRR) since the NPV is a monotone decreasing function of r .
- If $F_0 < 0$ but $F_t > 0$ or $F_t < 0$ (change in signs of cash flows), there may exist numerous solutions since the NPV is not a monotone decreasing function of r .

Example: Many IRR:

Project	F_0	F_1	F_2	NPV (r=10%)	IRR
C	-4 000	25 000	-25 000	-1 933	25% et 400%

The NPV of this stream of cash flows has the structure of a quadratic equation : $-4r^2 + 17r - 4 = 0$. There exists two solutions to this equation: 0,25 et 4. The NPV is a concave function of r .

Example: No IRR:

Project	F_0	F_1	F_2	NPV (r=10%)	IRR
D	1 000	-3 000	2 500	339	—

The NPV is a convex function of r and does not cross the horizontal axis

d) **The IRR cannot take into account the term structure of interest rate**

3.6 NPV and Inflation

The relationship between nominal (F) and real (f) cash flows is given by:

$$f_t = \frac{F_t}{(1 + E(\pi))^t}$$

with $E(\pi)$: expected inflation

In the same fashion, the relationship between the nominal (r) and the real interest rate (r_r) is given by:

$$(1 + r) = (1 + r_r)(1 + E(\pi))$$

or,

$$r_r = [(1 + r)/(1 + E(\pi))] - 1$$

Example:

- The project's nominal cash flows are the following:
 - -1 000 at $t=0$
 - 400 per year for $t = 1-5$
- Cost of capital (nominal) = 10%
- Expected annual inflation rate: 5%

The NPV of the project is then given by:

$$V = \sum_{t=0}^5 \frac{F_t}{(1 + r)^t}$$

NPV in nominal

$$NPV = -1\,000 + \frac{400}{(1.10)} + \frac{400}{(1.10)^2} + \frac{400}{(1.10)^3} + \frac{400}{(1.10)^4} + \frac{400}{(1.10)^5} = 516.31$$

NPV in real

We need to calculate the real cash flows and interest rate.

$$\begin{aligned} f_0 &= F_0/(1 + E(\pi)) = -1\,000 \\ f_1 &= F_1/(1 + E(\pi))^1 = 400/(1.05) = 380.95 \\ f_2 &= F_2/(1 + E(\pi))^2 = 400/(1.05)^2 = 362.82 \\ f_3 &= F_3/(1 + E(\pi))^3 = 400/(1.05)^3 = 345.54 \\ f_4 &= F_4/(1 + E(\pi))^4 = 400/(1.05)^4 = 329.08 \\ f_5 &= F_5/(1 + E(\pi))^5 = 400/(1.05)^5 = 313.41 \end{aligned}$$

In addition,

$$r_r = \frac{(1 + r)}{(1 + E(\pi))} - 1 = \frac{1.1}{1.05} - 1 = 0.0476$$

$$NPV = -1\,000 + \frac{380.95}{(1.0476)} + \frac{362.82}{(1.0476)^2} + \frac{345.54}{(1.0476)^3} + \frac{329.08}{(1.0476)^4} + \frac{313.41}{(1.0476)^5} = 516.31$$

Conclusions:

- Inflation is neutral on the value of an asset or a project if all the cash flows are indexed on inflation.
- Problem: Generally, this is not the case since some of the firm's cash flows are not indexed on inflation.

3.7 Inflation, Depreciation and Valuation

The value of an asset or an investment project is equal to the present cash flow generated by the asset or the project. We also know that the assets are depreciated, either on a linear or declining basis. Although depreciation does not represent a cash flow, the tax saving associated to depreciation is an inflow which needs to be taken into consideration when valuing an asset. In this context, the presence of inflation will impact on the valuation since depreciation and the associated tax saving are based on book value and hence, not indexed on inflation.

Example:

- Purchasing cost of a new machine: \$2 400 000
- Linear depreciation over 4 years
- Annual depreciation: \$ 600 000
- Annual increase in EBITDA with the new machine: \$1 000 000 (real)
- Tax rate: 50%.
- Cost of capital (real): 8%

Case 1: No inflation

t	CAPEX	EBITDA(1 - τ)	$\tau \times$ DEP	CF
0	-2 400 000	—	—	-2 400 000
1		500 000	300 000	800 000
2		500 000	300 000	800 000
3		500 000	300 000	800 000
4		500 000	300 000	800 000

$$NPV = -2\,400\,000 + \frac{800\,000}{0.08} \left[1 - \frac{1}{(1.08)^4} \right] = 249\,702$$

Case 2: Inflation = 4% per year

t	CAPEX	EBITDA(1 - τ)	$\tau \times$ DEP	CF
0	-2 400 000	—	—	-2 400 000
1		520 000	300 000	820 000
2		540 800	300 000	840 800
3		562 432	300 000	862 432
4		584 929	300 000	884 929

Cost of capital (nominal) = $(1.08) \times (1.04) - 1 = 0.1232 = 12.32\%$

$$NPV(\text{nominal}) = -2\,400\,000 + \frac{820\,000}{1.1232} + \frac{840\,800}{(1.1232)^2} + \frac{862\,432}{(1.1232)^3} + \frac{884\,929}{(1.1232)^4} = 161\,161$$

Impact of inflation: Reduces the NPV of a project or the PV value of an asset.

3.8 NPV vs. IRR: Empirical Evidence

Graham et Harvey (1999):

- 75% of firms always or almost always use the NPV for capital budgeting.
- 76% of firms always or almost always use the IRR for capital budgeting.
- The use of NPV is more frequent in large than small firms.
- Small firms rely more on the PAYBACK than NPV and IRR.
- PAYBACK is more popular for older managers without MBA studies.

Conclusions:

1. The NPV is commonly used by firms for their capital budgeting decision.
2. Firms use many criteria simultaneously since each of these provides different information.

4 Fundamental Notions on Risk

Let's consider the behavior of agents in the context of uncertainty, in particular in their choice of risky assets. The analysis under uncertainty applies not only to financial assets but to a large number of decisions such as the purchase and financing of a house, the decision to undertake university studies or a specific career, ...

4.1 General notions

In order to analyze the behavior of agents under uncertainty, we will use the notion of **gamble** and a number of assumptions on the behavior of rational agents.

First, let's define a general decision rule.

“A *gamble* $\tilde{X}$ is preferred to a gamble $\tilde{Y}$ if the value attached to $\tilde{X}$, $V(\tilde{X})$, is greater than the value attached to $\tilde{Y}$, $V(\tilde{Y})$ ”.

Mathematically, this can be written as:

$$\tilde{X} \succ \tilde{Y} \iff V(\tilde{X}) > V(\tilde{Y})$$

where $\succ$ indicates a preference relationship.

Def.: A **gamble** is a set $[X, P]$ where:

- $X = (x_1, x_2, x_3, \dots, x_n)$ is a vector of quantity goods to be received in each state of nature.
- $P = (p_1, p_2, p_3, \dots, p_n)$ is a vector of probability of each state of nature with $\sum_{s=1}^n p_s = 1$.
- A state of nature is a complete description of the environment that the agent is facing.

Consider an agent with an initial wealth of W_0 facing a gamble $\tilde{X}$.

- If the gamble is **additive**, his final wealth is equal to: $\widetilde{W}_f = W_0 + \tilde{X}$.
- If the gamble is **multiplicative**, his final wealth is equal to: $\widetilde{W}_f = W_0(1 + \tilde{X})$.
- The agent's decision will be centered on his final wealth which depends on the risk surrounding the gamble.
- We will assume that the gamble is additive for the rest of the presentation.

Example: Consider a game which consists in rolling a dice. If you roll # 1, you win \$12, otherwise you lose. The price for playing this game is \$2. This gamble can be depicted as follows:

$[X, P] = [-2, 10; 5/6, 1/6]$, that is you have 5 chances out of 6 to lose \$2 and 1 chance out of 6 to win \$10 (net gain if you win).

Another alternative would be not to play, in which case the gamble would be: $[X, P] = [0; 1]$, that is you lose nothing with a probability equal to 1.

Prediction: If an agent is risk averse, he will prefer not to play this gamble. See later in the § Risk Aversion

Question: How do we value a gamble? What is what is the value function behind $V(\tilde{W}_f)$?

Answer: Consider the following three criteria:

1. **Arithmetic mean:** $V(\tilde{W}_f) = E(\tilde{W}_f) = E(W_0 + \tilde{X}) = W_0 + E(\tilde{X})$

Consider the following three gambles:

X	Prob(X)	Y	Prob(Y)	Z	Prob(Z)
9	0.5	8	0.5	9	0.5
11	0.5	12	0.5	13	0.5

If we use the arithmetic mean criteria to value these gambles, we get:

- $E(\tilde{X}) = 0.5(9+11) = 10$
- $E(\tilde{Y}) = 0.5(8+12) = 10$
- $E(\tilde{Z}) = 0.5(9+13) = 11$

Therefore, $\tilde{Z} \succ \tilde{X} \sim \tilde{Y}$

This approach suffers from two problems:

- (a) Does not account for the risk of an asset. It is based only on the first moment of the distribution without considering higher order moments such as the variance.
- (b) St-Petersburg Paradox (Bernoulli, 1713)

- Consider a game where an agent can toss a coin (head or tail).
- The agent keeps playing until he flips *Head*.
- He wins 2^n if he gets *Head* on the n^{th} toss. This happens with probability $(1/2)^n$.
- The player can play as often as he wants.

Question: How much would an agent be willing to pay to play this game?

Answer: If the agent uses the arithmetic mean criteria, the value of this game is equal:

$$E(\text{Gain}) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n 2^n = +\infty$$

When asked how much someone would be willing to pay for this game, the amount was finite. This suggests that agents are not using the arithmetic mean criteria or using

other considerations to value risky gambles.

As we will see later, the use of the arithmetic criteria corresponds to the assumption of risk neutrality.

2. Mean-Variance: $V(\tilde{W}_f) = g(E(\tilde{W}_f), \sigma^2(\tilde{W}_f))$

The mean-variance criteria is a response to the first critique of the arithmetic mean criteria whereby the risk of an asset (gamble) is captured by its variance.

Let $\tilde{W}_f = W_0 + \tilde{X}$. We can then write that the value of a gamble is equal to:

$$V(\tilde{W}_f) = g(E(\tilde{W}_f), \sigma^2(\tilde{W}_f))$$

This expression means that the value of a risky final wealth is a function of its expected value, $E(\tilde{W})$, and its variance, $\sigma^2(\tilde{W})$.

Question: What are the signs of g_1 and g_2 ?

$$g_1 = \left. \frac{\partial V_i(\cdot)}{\partial E(\tilde{X})} \right|_{\sigma^2=c} > 0$$

Interpretation: For a given level of risk (variance), the value of a gamble increases with its expected payoff. An agent will always prefer more wealth to less wealth (for a given risk) irrespective of its risk aversion.

$$g_2 = \left. \frac{\partial V_i(\cdot)}{\partial \sigma^2(\tilde{X})} \right|_{E(\tilde{X})=c} = ?$$

- $g_2 < 0$: $\uparrow$ risk $\Rightarrow$ $\downarrow$ value of the gamble: **risk aversion**
- $g_2 > 0$: $\uparrow$ risk $\Rightarrow$ $\uparrow$ value of the gamble: **risk loving**
- $g_2 = 0$: $\uparrow$ risk $\Rightarrow$ no impact on value of the gamble: **risk neutral**

The value function $V(\tilde{W}_f) = g(E(\tilde{W}_f), \sigma^2(\tilde{W}_f))$ can be seen as a utility function with indifference curves in the space $[E(\tilde{W}_f), \sigma^2(\tilde{W}_f)]$ (See §5).

3. Expected utility: $V(\tilde{W}_f) = E[U(\tilde{W}_f)] = \sum_{s=1}^n p_s u(W_0 + \tilde{X}_s)$

- The expected utility criteria is a solution to the above two critiques.
- It can take into account the risk of the assets and the fact that agents are not valuing 1€ as 1€ but rather as the utility derived from 1€, $U(1)$.
- It is assumed that $U'(\tilde{W}) > 0$, that is the marginal utility of wealth is positive.

Let's illustrate these valuation criteria with games.

- A TV game invites 5 participants to play a game with the chance of winning different prizes ranging from 20€ to 5 000€. You are one of these participants.
- Once each participant has played its game, they are ranked from the highest to the lowest prize winner.
- The highest winning prize is 3 000, the second, 2 000, etc.
- The highest winner is offered two possibilities:
 1. Leave the show and keep your 3 000.
 2. Trade the 3 000 against a game where there are different prizes hidden in 4 TV screens with a big prize of 15 000. You know the value of each prize but you don't know in which screen they are. The prizes are:

20	3 000
5 000	15 000

- You can win 20, 3 000, 5 000 or 15 000 with 1/4 chance.
- Question: would you be willing to trade your prize of 3 000 for this game?

To answer this question, we need to value each alternative using a value function.

1. Arithmetic mean, $V(\tilde{W}) = E(\tilde{W})$

Alternative 1: Keep the money.

$$V(\tilde{W}_f) = E(\tilde{W}_f) = 3\,000$$

Alternative 2: Play the game with the TV screens.

$$V(\tilde{W}_f) = E(\tilde{W}_f) = \frac{1}{4}(20 + 3\,000 + 5\,000 + 15\,000) = 5\,755$$

Based on the arithmetic mean criteria, you will decide to give up your 3 000 and play the game with the TV screen.

2. Expected utility, $V(\tilde{W}) = E[U(\tilde{W})]$ with $U(W) = \ln(W)$.

Alternative 1: Keep the money.

$$V(\tilde{W}_f) = U(3\,000) = \ln(3\,000) = 8.01$$

Alternative 2: Play the game with the TV screens.

$$\begin{aligned} V(\tilde{W}_f) &= E[U(\tilde{W}_f)] \\ &= \frac{1}{4}(U(20) + U(3\,000) + U(5\,000) + U(15\,000)) \\ &= \frac{1}{4}(\ln(20) + \ln(3\,000) + \ln(5\,000) + \ln(15\,000)) \\ &= 7.28 \end{aligned}$$

Based on the expected utility criteria with a logarithmic utility function, you will keep your 3 000.

This game was a very famous game on French TV (many years ago and TODAY)!!!!!!

Conclusion: The decision of agents facing risky situations depends on their attitude towards risk which is captured by the utility function. As we will see in the next section, the first scenario captures **risk neutral** behavior while the second scenario captures **risk averse** behavior.

Risk Analysis: Application 1 (In class)

- A few weeks ago, you turned on the TV and felt on a famous TV show called “The Banker” where the objective is to make it to the end of the game with the box containing the highest prize among 25 boxes. At the time you joined the show, the participant’s situation was the following:

1. There are 6 boxes left with prizes of 1, 25, 50, 500, 25 000 and 100 000
2. Each prize have equal probability: 1/6
3. The participants has one box in front of him without knowing the prize in the box

Suddenly, the phone on the table rings. It is THE BANKER. He proposes to the participant the following deal:

1. Option 1: He can continue the game to the end and hope to finish with the highest prize
2. Option 2: He can stop the game now and leave with 7 999€

The host announces that they will be back after the advertizing at which time the participant must announce his decision.

What would you do?

- Question: Evaluate this decision using the expected utility criteria with the following three utility functions:
 - $U(\tilde{W}) = E(\tilde{W})$
 - $U(\tilde{W}) = \ln(\tilde{W})$
 - $U(\tilde{W}) = \tilde{W}^2$

4.2 Attitude towards risk

Using the expected utility criteria, we can now define an agent’s attitude towards risk.

Def.: A gamble is said to be **fair** if its expected gain is zero.

Using the notion of fair gamble, we can give a first definition of risk aversion.

Def.: An agent is **risk averse** if he refuses to participate to a fair gamble.

Let W_0 be the initial wealth of an agent who face a gamble $(1/2, 1/2; -X, +X) \quad \forall X \in [0, W_0]$. We can write that an agent is risk averse if the following condition is satisfied:

$$U(W_0 + E(\tilde{x})) > \frac{1}{2}U(W_0 - X) + \frac{1}{2}U(W_0 + X)$$

with $E(\tilde{X}) = 0$

The lefthand side of the equation is the utility of the expected wealth while the righthand side is the expected utility of the random wealth. This means that a risk averse agent prefers to receive with certainty the expected gain of a fair gamble than facing the gamble itself.

Let's come back on the example at seen earlier.

- Consider a game which consists in rolling a dice.
- If you roll $\neq 1$, you win \$12, otherwise you lose.
- The price for playing this game is \$2.
- Gamble: $[X, P] = [-2, 10; 5/6, 1/6]$, that is you have 5 chances out of 6 to lose \$2 and 1 chance out of 6 to win \$10 (net gain if you win).
- Another alternative would be not to play, in which case the gamble would be: $[X, P] = [0; 1]$, that is you lose nothing with probability equal to 1.
- Prediction: If an agent is risk averse, he will prefer not to play this gamble.

Explanation

- This game is a fair gamble.
- $E(\tilde{X}) = 5/6(-2) + 1/6(10) = 0$
- A risk averse agent will prefer 0 with certainty (not play) than 0 with uncertainty (play).

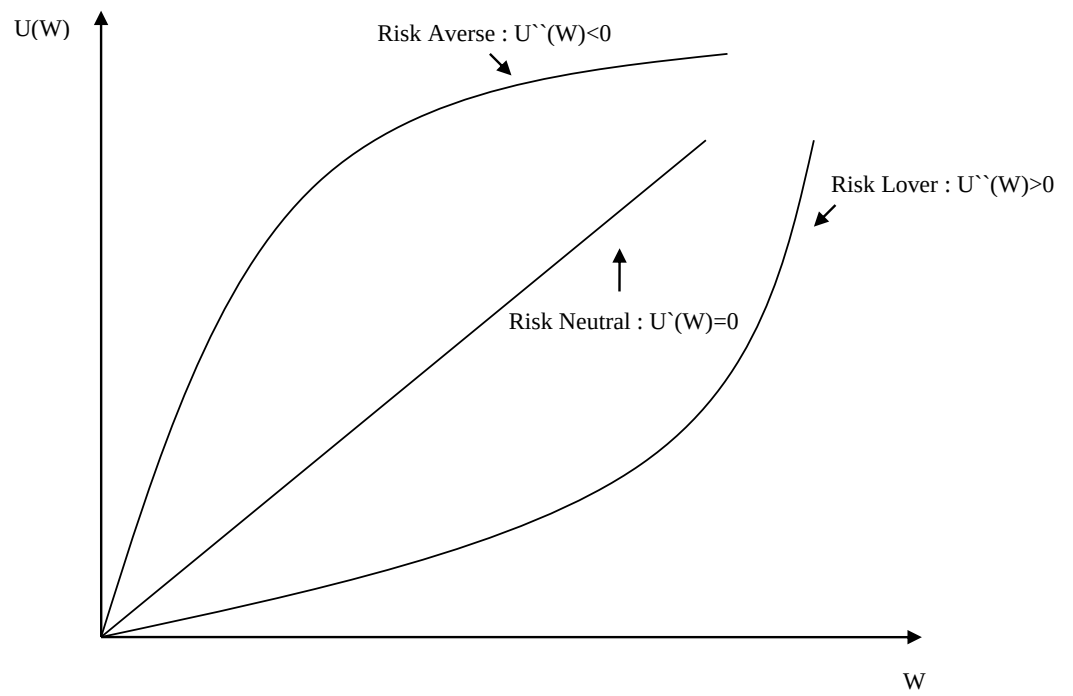
More generally, consider a gamble $\tilde{X}$ such that $E(\tilde{X}) = 0$ and $\sigma^2(\tilde{X}) > 0$. Thus, we can write,

- Risk averse: $U[E(\tilde{W}_f)] > E[U(\tilde{W}_f)]$ or $U[W_0 + E(\tilde{X})] > E[U(W_0 + \tilde{X})]$
- Risk neutral: $U[E(\tilde{W}_f)] = E[U(\tilde{W}_f)]$ or $U[W_0 + E(\tilde{X})] = E[U(W_0 + \tilde{X})]$
- Risk lover: $U[E(\tilde{W}_f)] < E[U(\tilde{W}_f)]$ or $U[W_0 + E(\tilde{X})] < E[U(W_0 + \tilde{X})]$

Risk aversion is captured by the shape of the utility function. Indeed,

Risk averse $\implies$ strictly **concave** utility function ($U''(\tilde{W}_f) < 0$)
Risk neutral $\implies$ **linear** utility function ($U''(\tilde{W}_f) = 0$)
Risk lover $\implies$ strictly **convex** utility function ($U''(\tilde{W}_f) > 0$)

The following graph illustrates the utility function for different attitudes towards risk.



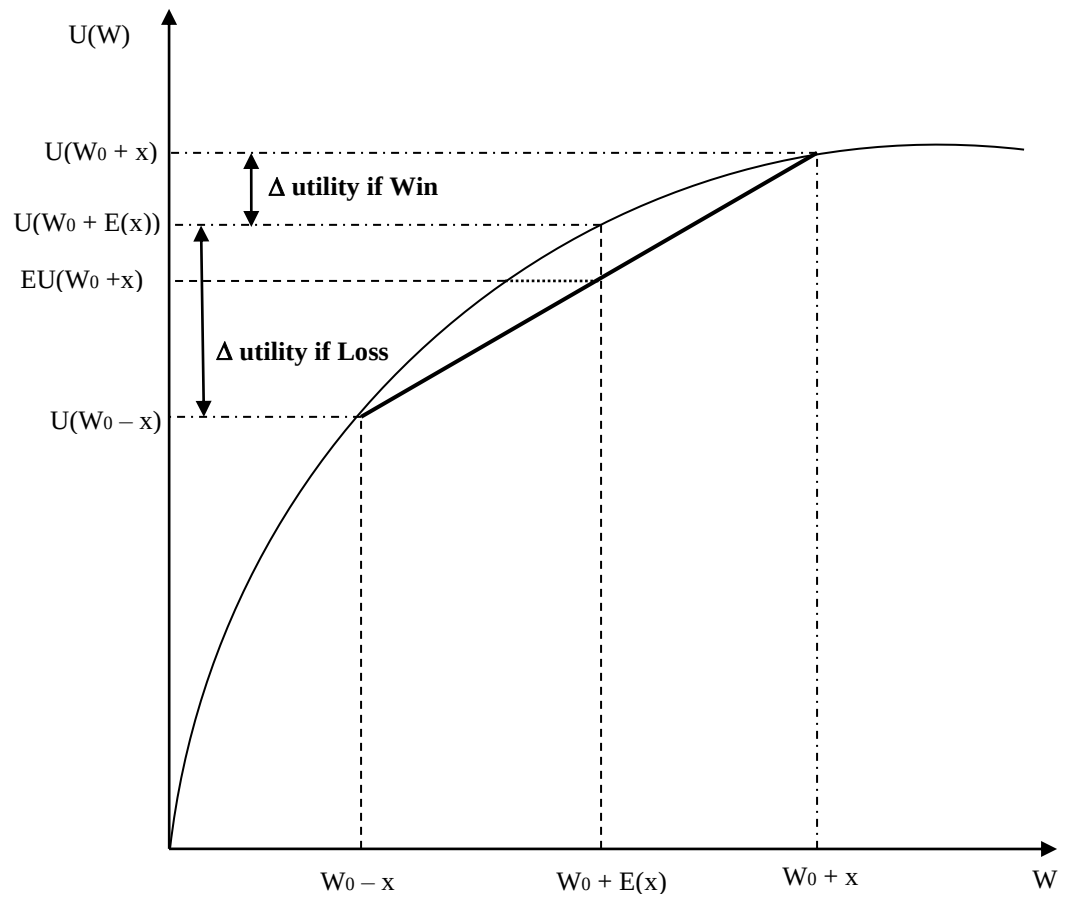
Let us reconsider the fair gamble $(1/2, 1/2; -X, +X)$. We have seen that if an agent is risk averse, we can write:

$$\begin{aligned}
 U[E(\tilde{W}_f)] &> E[U(\tilde{W}_f)] \\
 U\left(\frac{1}{2}(W_0 - X) + \frac{1}{2}(W_0 + X)\right) &> \frac{1}{2}U(W_0 - X) + \frac{1}{2}U(W_0 + X) \\
 U(W_0) &> \frac{1}{2}U(W_0 - X) + \frac{1}{2}U(W_0 + X) \\
 2U(W_0) &> U(W_0 - X) + U(W_0 + X) \\
 U(W_0) - U(W_0 - X) &> U(W_0 + X) - U(W_0)
 \end{aligned}$$

This condition says that for a risk averse agent, the variation in utility associated to a loss relative to the expected wealth (lefthand side) is larger than the variation in utility associated to a win relative to the expected wealth (righthand side). Therefore, losing 100 and winning 100 is symmetric in variation of wealth but not in variation of utility. This reflects the concavity of the utility function.

The following graphic depicts the utility function of a risk averse agent.

Utility Function : Risk Aversion



4.3 Certainty Equivalent, Insurance Premium & Risk Premium

We know that $\tilde{W}_f = W_0 + \tilde{X}$ and that agents are maximizing their expected utility.

$$E[U(\tilde{W}_f)] = \sum_{s=1}^n p_s u(W_0 + \tilde{X}_s)$$

Facing a risky situation, an agent can choose between taking the risk or withdrawing from that risk. This brings us to a first concept linked to the attitude towards risk, that is the concept of certainty equivalent.

Def: The **certainty equivalent**, W^* , is the amount of final wealth with certainty which provides the same level of utility as the final wealth with the gamble (with uncertainty), $W_0 + \tilde{X}$.

W^* is such that:

$$U[W^*] = E[U(\tilde{W}_f)] = E[U(W_0 + \tilde{X})]$$

Thus, we can write:

$$W^* = U^{-1}[E[U(\tilde{W}_f)]]$$

W^* is such that the agent is indifferent between W^* (without the gamble) and $\tilde{W}_f$ (with the gamble).

A second well-known concept related to attitude towards risk is the concept of insurance premium.

Def: The **insurance premium** (price of a gamble), I , is the minimum amount of money (price) required by an agent to get rid of a gamble.

I is such that:

$$U[W_0 + I] = E[U(\tilde{W}_f)]$$

Theorem:

1. Risk averse ($U'' < 0$): the insurance premium (I) is lower than the expected gain of the gamble, $I < E(\tilde{X})$.
2. Risk neutral ($U'' = 0$): the insurance premium (I) is equal than the expected gain of the gamble, $I = E(\tilde{X})$.
3. Risk lover ($U'' > 0$): the insurance premium (I) is greater than the expected gain of the gamble, $I > E(\tilde{X})$.

Proof (risk aversion):

We know that I is such that $U[W_0 + I] = E[U(W_0 + \tilde{X})]$

For a risk averse agent, $U[W_0 + E(\tilde{X})] > E[U(W_0 + \tilde{X})]$

$$\begin{aligned}
\Rightarrow & U[W_0 + E(\tilde{X})] > U[W_0 + I] \\
\Rightarrow & W_0 + E(\tilde{X}) > W_0 + I \\
\Rightarrow & E(\tilde{X}) > I
\end{aligned}$$

Note: In the context of a classic insurance problem, $E(\tilde{X})$ and I are negative numbers which means that a risk averse agent is always willing to pay more than the expected loss in order to get insured.

Finally, a third concept related to attitude towards risk is the concept of risk premium.

Def: The **risk premium**, Π , is the maximum wealth an agent would be willing to pay in order to get rid of the risk and receive with certainty the expected gain of the gamble and be indifferent between taking and not taking the risk.

Π is such that:

$$U[W_0 + E(\tilde{X}) - \Pi] = E[U(\tilde{W}_f)]$$

Theorem:

1. Risk averse ($U'' < 0$): $\Pi > 0$
2. Risk neutral ($U'' = 0$): $\Pi = 0$
3. Risk lover ($U'' > 0$): $\Pi < 0$

Proof (risk aversion):

We know that Π is such that $U[W_0 + E(\tilde{X}) - \Pi] = E[U(W_0 + \tilde{X})]$

For a risk averse agent, $U[W_0 + E(\tilde{X})] > E[U(W_0 + \tilde{X})]$

$$\begin{aligned}
\Rightarrow & U[W_0 + E(\tilde{X})] > U[W_0 + E(\tilde{X}) - \Pi] \\
\Rightarrow & W_0 + E(\tilde{X}) > W_0 + E(\tilde{X}) - \Pi \\
\Rightarrow & \Pi > 0
\end{aligned}$$

Summary table for attitude towards risk:

	Risk averse	Risk neutral	Risk lover
Utility function	$U'' < 0$	$U'' = 0$	$U'' > 0$
Insurance premium	$E(\tilde{X}) > I$	$E(\tilde{X}) = I$	$E(\tilde{X}) < I$
Risk premium	$\Pi > 0$	$\Pi = 0$	$\Pi < 0$

As one can see, these concepts are all linked through the expected utility criteria. Indeed:

$$\begin{aligned}
U[W^*] &= E[U(\tilde{W}_f)] \\
U[W_0 + I] &= E[U(\tilde{W}_f)] \\
U[W_0 + E(\tilde{X}) - \Pi] &= E[U(\tilde{W}_f)]
\end{aligned}$$

Thus, we can get that:

$$\Pi = E(\tilde{X}) - I$$

or

$$\Pi = W_0 + E(\tilde{X}) - W^*$$

or

$$I = W^* - W_0$$

We can measure risk aversion:

- **Absolute Risk Aversion** (ARA) = $-\frac{U''(W_0)}{U'(W_0)}$

Assumption:

$$\text{H1 : } \frac{\partial \text{ARA}}{\partial W} \leq 0$$

Example: Let $W_0 = 500$, $U(W) = \ln(W)$ and the gamble $[X, P] = [-50, 0; 0.2, 0.8]$

a) **Certainty equivalent:** W^* is such that $U[W^*] = E[U(\tilde{W}_f)] = E[U(W_0 + \tilde{X})]$

Step 1: Calculate the wealth with the gamble in each state of nature.

Prob	$\tilde{W}_f$
0.2	450
0.8	500

Step 2: Calculate the expected utility of wealth

$$E[U(\tilde{W}_f)] = 0.2 \ln(450) + 0.8 \ln(500) = 6.194$$

Step 3: Calculate the certainty equivalent. The inverse function of the $\ln(\cdot)$ function is the exponential function. Thus,

$$W^* = U^{-1}[E[U(\tilde{W}_f)]] = e^{6.194} = 489.57$$

This means that the agent is indifferent between having 489.57 with certainty and facing the gamble and having 450 with a 20% chance and 500 with a 80% chance.

b) **Insurance premium:** I is such that $U[W_0 + I] = E[U(\tilde{W}_f)]$

As we know, $I = W^* - W_0 = 489.57 - 500 = -10.43$. This is the price that the agent is willing to pay to get rid of the risk.

One can verify the condition that $E(\tilde{X}) > I$ since $E(\tilde{X}) = 0.2(-50) = -10$.

c) **Risk premium:** Π is such that $U[W_0 + E(\tilde{X}) - \Pi] = E[U(\tilde{W}_f)]$

As we have seen, $\Pi = E(\tilde{X}) - I = -10 + 10.43 = 0.43$. This means that the agent is willing to pay more (10.43) than the expected loss of the gamble (10) in order to get insured. The difference is equal to the risk premium and derives from risk aversion.

Risk Analysis: Application 2 (In class)

Let $W_0 = 150$, $U(W) = (W)^{1/2}$ and the gamble $[X, P] = [-80, 40, 120; 0.3, 0.4, 0.3]$

Calculate the certainty equivalent and the insurance premium for this agent. How do you interpret these results?

Example: Job offer

Imagine that you work in an investment bank and your salary is 100 000 €. You just received a job offer from a start-up where your salary would depend on the firm's performances. Your salary could be 60 000 € with 50% probability or 150 000 € with 50% probability.

Suppose that you are risk averse and that your utility function is represented by $U(W) = \ln(W)$. Should you accept this new offer? (One should note that this is not a fair gamble!).

First, let's calculate the utility associated to each salary level.

- $W = 60\,000 \implies U(W) = \ln(60\,000) = 11.002$
- $W = 100\,000 \implies U(W) = \ln(100\,000) = 11.510$
- $W = 150\,000 \implies U(W) = \ln(150\,000) = 11.918$

To determine whether or not this new job offer is interesting, you need to compare the utility derived from your actual salary and the expected utility derived from the salary in your new position which is equal to:

$$\begin{aligned} E[U(\tilde{W}_f)] &= 0.5U(60\,000) + 0.5U(150\,000) \\ &= (0.5 \times \ln(60\,000)) + (0.5 \times \ln(150\,000)) \\ &= 11.46025 \end{aligned}$$

Thus, if you have the choice between your actual position with a utility level of 11.513 and your new position with an expected utility level of 11.46, you will keep your actual position. In other words,

$$U[W_f] > E[U(\tilde{W}_f)] \Rightarrow U(100\,000) > 0.5U(60\,000) + 0.5U(150\,000)$$

Note: One can see that relative to your actual salary (100 000), the variation in utility associated to a decrease in salary (60 000) is equal to 0.511 while the variation in utility associated to an increase in salary (150 000) is equal to 0.405.

Question: What should have been your actual salary level for the proposition to be interesting?

The answer to this question is the certainty equivalent which is the level of wealth with certainty that would generate the same level of utility as the wealth with uncertainty (new job). This is equal to:

$$W^* = U^{-1}[E[U(\tilde{W}_f)]] = e^{11.46025} = 94\,868.33$$

Thus, you would have been indifferent between your actual job and the job offer if your salary was 94 868.33. Your actual salary would have to be lower than this amount for the new position to be interesting.

The expected utility criteria requires the use of a utility function. The choice of this function depends on the properties we believe prevail for an agent (i.e. risk aversion). The most common utility functions are:

1. **Affine:** $U(W) = a + bW \Rightarrow U'' = 0$ (risk neutrality); H1 satisfied.
2. **Logarithmic:** $U(W) = a + b \ln(W)$ ($b > 0$) $\Rightarrow U'' < 0$ (risk aversion); H1 satisfied.
3. **Quadratic:** $U(W) = a + bW - cW^2 \Rightarrow U'' \leq$ or ≥ 0 (depends on the sign of c); H1 not satisfied.
4. **Negative exponential (CARA):** $U(W) = -e^{-aW}$ ($a > 0$) $\Rightarrow U'' < 0$ (risk aversion); H1 satisfied.
5. **Power (CRRA):** $U(W) = \frac{W^b}{b}$ ($b < 1$) $\Rightarrow U'' < 0$ (risk aversion); H1 satisfied.
6. **Mean - Variance:** $U(W) = f(E(W), \sigma^2(W))$ with $\frac{\partial U(W)}{\partial E(W)} > 0$ and $\frac{\partial U(W)}{\partial \sigma^2(W)} < 0$

2 main properties of the negative exponential utility function:

1. it represents the same preferences as the Markowitz utility function $[E(W), \sigma^2(W)]$ defined on the mean and variance of final wealth;
2. the risk premium is exact and valid for any gamble and not only for gambles with small risk.

4.4 Increasing Risk and Variance

Question: Can we predict the choice of agents among risky assets on the basis of partial information on their preferences?

In the context of financial assets, under what condition can we say that some assets are preferred to others by individual agents if the only information available on the preferences of these agents is that: i) $U' > 0$ or ii) $U'' < 0$?

In order to answer this question, we need more advanced criteria such as **stochastic dominance** criteria. But let's consider one criterion which is called a "mean-preserving spread".

Consider two random wealths, $\widetilde{W}$ and $\widetilde{W}'$, with a different distribution but the same expected value, $E(\widetilde{W}) = E(\widetilde{W}')$.

If agents are risk averse, we can say that $\widetilde{W}$ is preferred to $\widetilde{W}'$ if $\widetilde{W}$ leads to less risk than $\widetilde{W}'$. Alternatively, we can say that $\widetilde{W}$ is less risky than $\widetilde{W}'$ if it is preferred by all risk averse agents.

One way to express this is to write the following two equivalent conditions:

$$1) \ E[U(\widetilde{W})] > E[U(\widetilde{W}')] \quad \forall \ U'' < 0$$

$$2) \ \widetilde{W}' =^d \widetilde{W} + \epsilon \quad \text{with} \quad E(\epsilon/\widetilde{W}) = 0 \quad (\epsilon = \text{white noise})$$

Condition (2) says that $\widetilde{W}'$ is a **mean-preserving spread** of $\widetilde{W}$. Hence $\widetilde{W}'$ is more risky than $\widetilde{W}$.

As one can see, these two equivalent conditions do not rely on the use of the variance to determine the choice of individual agents among risky assets. Hence, how can we relate the variance to the risk.

There exists two propositions:

Proposition 1. A greater risk as defined by one of the equivalent conditions involves a greater variance.

Proof: Consider condition 2 on the distribution of wealth.

$$\begin{aligned} W' &=^d W + \epsilon \\ \Rightarrow \sigma^2(W') &= \sigma^2(W) + \sigma^2(\epsilon) \\ \Rightarrow \sigma^2(W') &> \sigma^2(W) \quad \text{since} \quad \sigma^2(\epsilon) > 0 \end{aligned}$$

Proposition 2. A greater variance does not necessarily involves a greater risk.

Let's illustrate proposition 2 with a numerical example.

1. Gamble X: [1, 5; 1/2, 1/2]
2. Gamble Y: [2, 8; 5/6, 1/6]

Thus,

$$E(X) = E(Y) = 3$$

$$\sigma^2(X) = 4 \quad \text{and} \quad \sigma^2(Y) = 5$$

Conclusion, if the variance is the appropriate measure of risk, $X \succ Y$ since it offers a lower risk for a same expected gain.

However, let's consider a more general criteria: the expected utility with $U(W) = \ln(W)$.

$$E[U(X)] = 0.5 \ln(1) + 0.5 \ln(5) = 0.8047$$

$$E[U(Y)] = 5/6 \ln(2) + 1/6 \ln(8) = 1.4241$$

Thus, $Y \succ X$ for all risk averse agents whose preferences are depicted by a logarithmic utility function even when Y has a greater variance than X . It must then be that the variance is not always the right measure of risk.

Conclusion: The use of the variance as a measure of risk requires an important assumption on the agent's preferences or the distribution of wealth.

1. Preferences: quadratic utility function: $U(W) = aW - bW^2$
2. Distribution of wealth: normal distribution: $W \sim N(\mu, \sigma^2)$

As we can see, the assumption on the preferences is violated by the logarithmic function.

5 Portfolio Theory: Mean-Variance Model

5.1 Risk and Return: Some Historical Facts and Figures

Question: What is the relationship between risk (volatility) and return?

Consider the following historical data for the U.S.

Volatility versus average return in the U.S., 1926–2014

Securities	Avg. Annual Return	Volatility (Standard Deviation)
Small stocks	18.8%	38.8%
S&P 500	12.0%	20.1%
Corporate Bonds	6.5%	7.0%
Treasury Bills	3.5%	3.1%

Source: Berk & DeMarzo, 2017

Conclusion: If variance is a good measure of risk or volatility, then there exists a positive relationship between the historical risk and return.

FIGURE 10.6

The Historical Trade-Off Between Risk and Return in Large Portfolios

Note the general increasing relationship between historical volatility and average return for these large portfolios. In addition to the portfolios in Figure 10.1, also included is a mid-cap portfolio composed of the 10% of U.S. stocks whose size is just above the median of all U.S. stocks. (Data from 1926–2014.)

Source: CRSP, Morgan Stanley Capital International

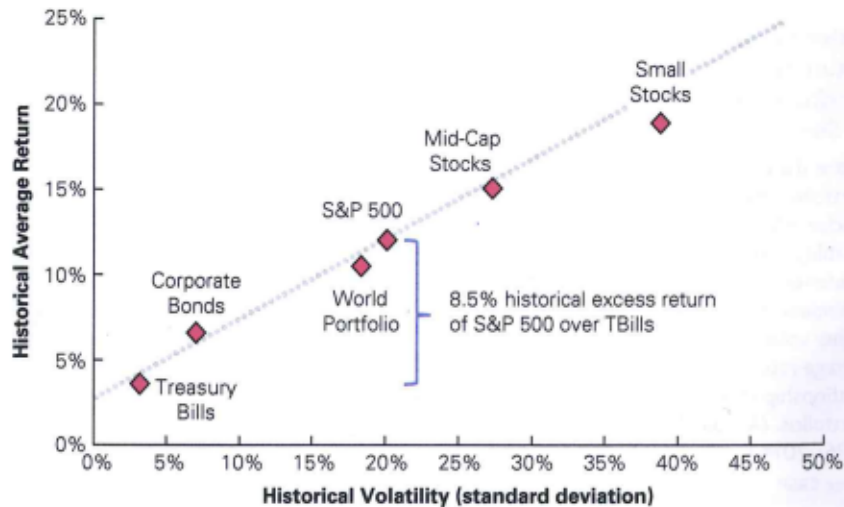
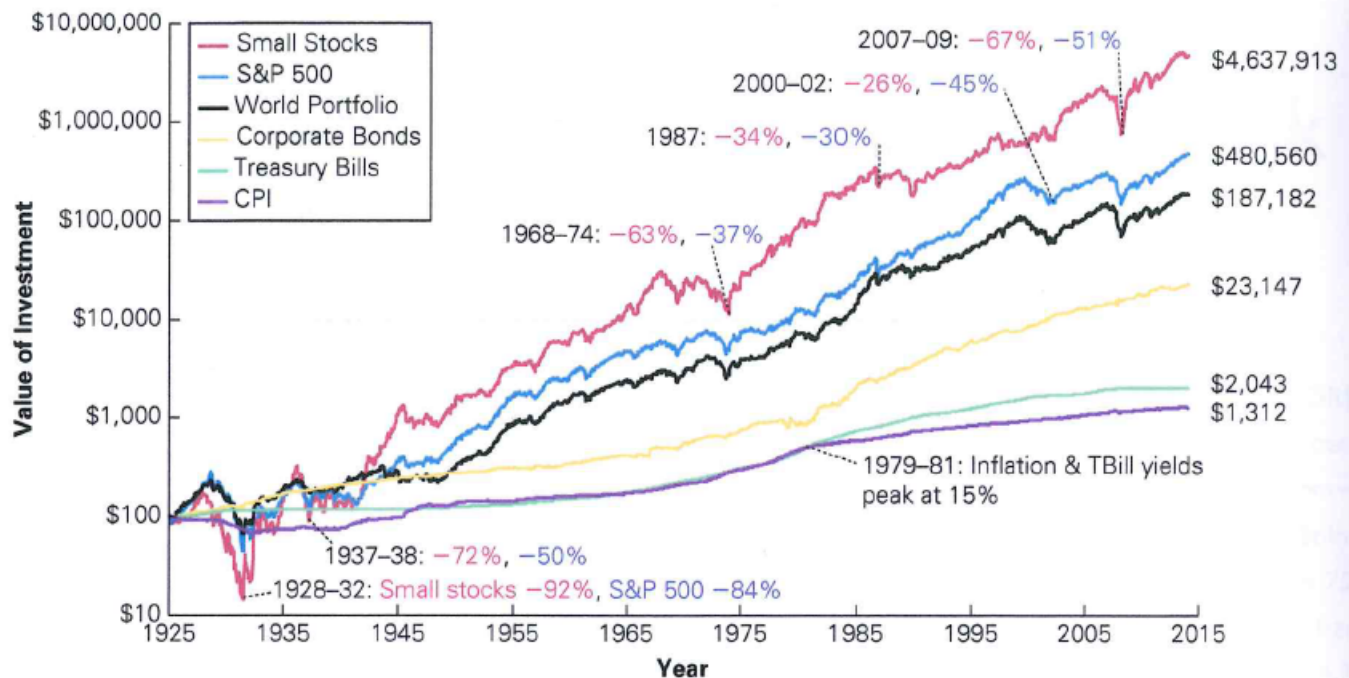


FIGURE 10.1
Value of \$100 Invested in 1925 in Stocks, Bonds, or Bills


The chart shows the growth in value of \$100 invested in 1925 if it were invested in U.S. large stocks, small stocks, world stocks, corporate bonds, or Treasury bills, with the level of the consumer price index (CPI) shown as a reference. Returns were calculated at year-end assuming all dividends and interest are reinvested and excluding transactions costs. Note that while stocks have generally outperformed bonds and bills, they have also endured periods of significant losses (numbers shown represent peak to trough decline, with the decline in small stocks in red and the S&P 500 in blue).

Source: Chicago Center for Research in Security Prices, Standard and Poor's, MSCI, and Global Financial Data.

5.2 The Mean - Variance Model

Assumptions:

1. No transaction costs.
2. Agents have homogeneous expectations on the return and risk of assets.
3. Risk averse agents choose their assets in order to maximize the expected utility defined over their final wealth, $\widetilde{W}_f = W_0(1 + \tilde{x})$.
4. The preferences of agent i can be represented by a utility function defined on the expected value and the variance (standard deviation) of a random wealth.

$$E[U_i(\widetilde{W}_f)] = E[U_i(W_0(1 + \tilde{X}))] = V_i[E(\tilde{X}), \sigma^2(\tilde{X})]$$

Properties of $V_i[E(\tilde{X}), \sigma^2(\tilde{X})]$

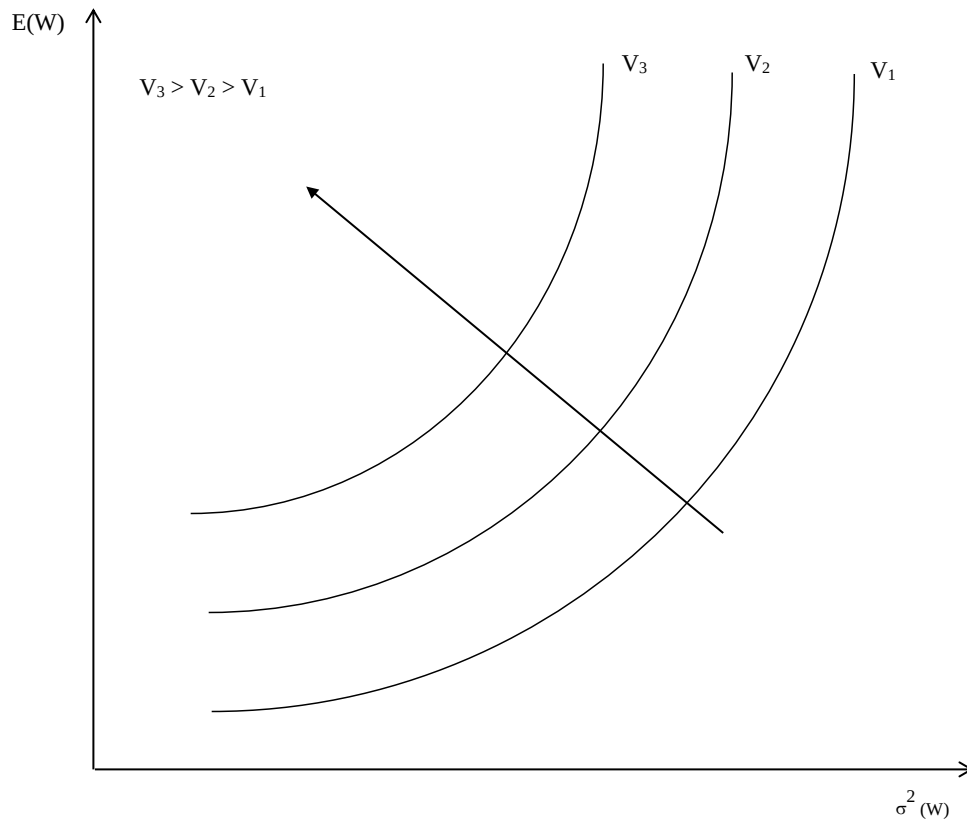
$$i) \quad \left. \frac{\partial V_i(\cdot)}{\partial E(\tilde{X})} \right|_{\sigma^2(\tilde{X})=c} > 0 \qquad ii) \quad \left. \frac{\partial V_i(\cdot)}{\partial \sigma^2(\tilde{X})} \right|_{E(\tilde{X})=c} < 0$$

Condition (i) indicates that for a given risk, the welfare of an agent increases in expected wealth (more wealth is preferred to less).

Condition (ii) says that for a given expected wealth or return, the welfare of an agent decreases with volatility (risk) of wealth (less risk is preferred to more risk). This holds for a risk averse agent.

From the utility function for a risk averse agent, we can derive indifference curves in the space $[E(\tilde{X}), \sigma^2(\tilde{X})]$.

Indifference curves : Mean-Variance



Notes:

- For a risk averse agent, indifference curves are upward sloping to reflect the fact that utility is increasing in expected wealth but decreasing in variance.
- An agent is willing to increase his risk (variance) if he is correctly compensated in terms of expected wealth in order to keep the same level of utility.
- The risk-return trade-off is given by the slope of the indifference curve. This is called the *Marginal Rate of Substitution*.
- For a risk neutral agent, the indifference curves are flat and horizontal. The level of utility only depends on the expected wealth and not on the variance.
- For a risk loving agent, the indifference curves are downward sloping to reflect the fact that utility is increasing in expected wealth and variance.

5.3 Economy with Risky Assets Only

Assumption: There exists n risky assets in the economy.

Def: A **portfolio** is a combination of assets: $(w_1, w_2, \dots, w_n)$

Def: The **weight** of an asset in a portfolio is equal to the proportion of wealth invested in the asset relative to the total wealth invested. Let P_i be the price of asset i and n_i be the number of shares of asset i , then:

$$w_i = \frac{n_i P_i}{\sum_{i=1}^n n_i P_i} \quad \text{with} \quad \sum_{i=1}^n w_i = 1$$

or

$$w_i = \frac{V_i}{\sum_{i=1}^n V_i}$$

with V_i equal to the market value of asset i in the portfolio.

Def: The **expected return** of a portfolio is equal to the weighted average of the expected return of each asset in the portfolio.

$$E(r_p) = \sum_{i=1}^n w_i E(r_i) = w_1 E(r_1) + w_2 E(r_2) + \dots + w_n E(r_n)$$

with

$$E(r_i) = \sum_s p_s r_{i,s} \quad \text{s: state of nature}$$

where,

- p_s : probability of state s
- $r_{i,s}$: return of asset i in state s

Def: The **variance** in the return of a portfolio is equal to:

$$\sigma^2(r_p) = E[(r_p - E(r_p))^2] = \sum_s p_s [r_{p,s} - E(r_p)]^2 \quad \text{with } p_s: \text{prob. of state } s$$

5.4 Portfolio with 2 Risky Assets

Consider 2 risky assets: 1 and 2.

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2)$$

$$\begin{aligned} \sigma^2(r_p) &= E[r_p - E(r_p)]^2 \\ &= E[w_1 r_1 + w_2 r_2 - w_1 E(r_1) - w_2 E(r_2)]^2 \\ &= E[w_1(r_1 - E(r_1)) + w_2(r_2 - E(r_2))]^2 \\ &= E[w_1^2(r_1 - E(r_1))^2 + w_2^2(r_2 - E(r_2))^2 + 2w_1 w_2(r_1 - E(r_1))(r_2 - E(r_2))] \\ &= w_1^2 E[(r_1 - E(r_1))^2] + w_2^2 E[(r_2 - E(r_2))^2] + 2w_1 w_2 E[(r_1 - E(r_1))(r_2 - E(r_2))] \\ &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \end{aligned}$$

with,

$$\sigma_{12} = \text{Cov}(r_1, r_2) = E[(r_1 - E(r_1))(r_2 - E(r_2))] = \sum_s p_s [r_{1,s} - E(r_1)][r_{2,s} - E(r_2)]$$

Knowing that:

$$\rho_{1,2} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

We can write:

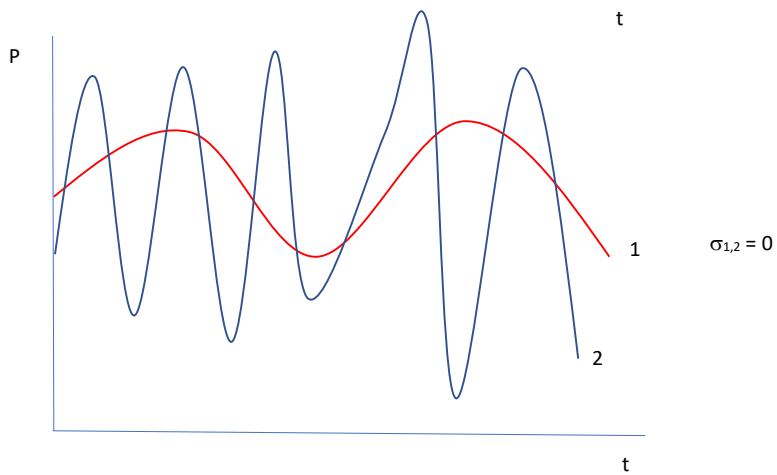
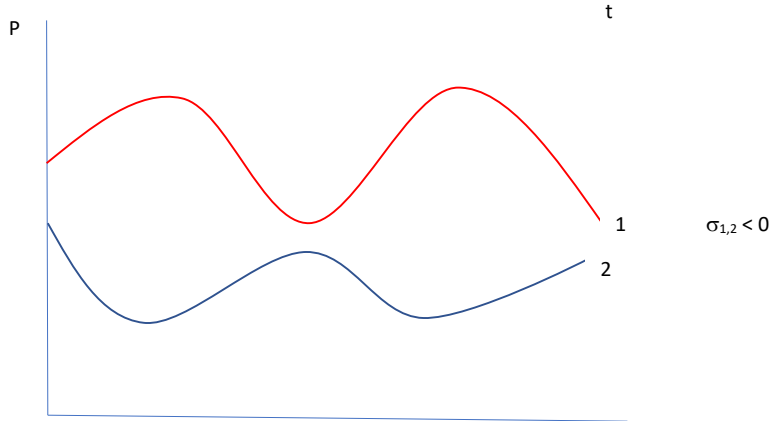
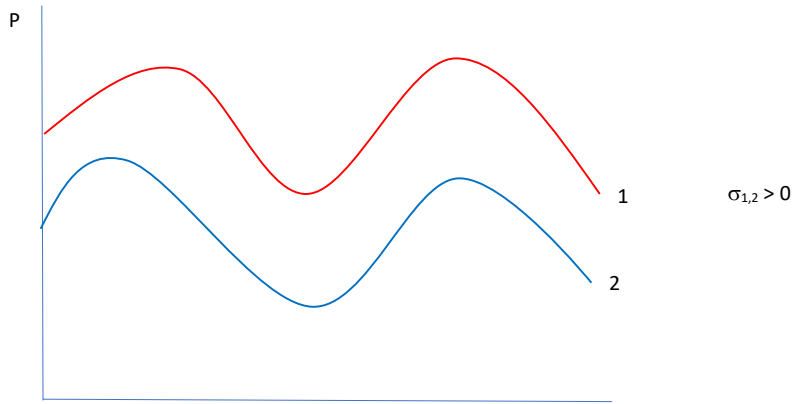
$$\sigma^2(r_p) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 \rho_{1,2}$$

The central concept in analyzing the risk of a portfolio is the **covariance**, σ_{12} .

- $\sigma_{12} < 0$: Returns of assets 1 and 2 vary in opposite directions
- $\sigma_{12} > 0$: Returns of assets 1 and 2 vary in the same direction
- $\sigma_{12} = 0$: Returns of assets 1 and 2 are linearly independent

The **coefficient of correlation**, $\rho_{i,j}$, is very useful since it $-1 \leq \rho_{i,j} \leq +1$ and has the same sign as the covariance.

Correlation with 2 assets



Example: Consider an economy with 3 assets and 5 states of nature.

State	Prob.	Return 1	Return 2	Return 3
A	0.2	-0.10	0.00	0.09
B	0.2	0.00	0.10	0.22
C	0.2	0.10	0.10	-0.06
D	0.2	0.20	0.15	-0.10
E	0.2	0.08	-0.06	0.25

First, let's examine the characteristics of each individual asset.

- Expected return

$$E(r_i) = \sum_s p_s r_{i,s}$$

$$E(r_1) = 0.2[-0.10 + 0 + 0.10 + 0.20 + 0.08] = 0.056$$

$$E(r_2) = 0.2[0 + 0.10 + 0.10 + 0.15 - 0.06] = 0.058$$

$$E(r_3) = 0.2[0.09 + 0.22 - 0.06 - 0.10 + 0.25] = 0.08$$

- Variance and standard deviation

$$\sigma^2(r_i) = E[r_i - E(r_i)]^2 = \sum_s p_s [r_{i,s} - E(r_i)]^2$$

$$\sigma^2(r_1) = 0.010144 \Rightarrow \sigma(r_1) = 0.1007$$

$$\sigma^2(r_2) = 0.005856 \Rightarrow \sigma(r_2) = 0.0765$$

$$\sigma^2(r_3) = 0.020212 \Rightarrow \sigma(r_3) = 0.1418$$

To measure the impact of diversification, let's create portfolios of 2 assets.

Portfolio I: $(w_1, w_3) = (0.5, 0.5)$

$$\begin{aligned} E(r_I) &= w_1 E(r_1) + w_3 E(r_3) \\ &= 0.5(0.056) + 0.5(0.08) \\ &= 0.068 \end{aligned}$$

and

$$\sigma^2(r_I) = w_1^2 \sigma_1^2 + w_3^2 \sigma_3^2 + 2w_1 w_3 \sigma_{13}$$

We need to calculate the covariance between assets 1 and 3.

$$\begin{aligned}
\sigma_{13} &= \sum_s p_s [r_{1,s} - E(r_1)][r_{3,s} - E(r_3)] \\
&= 0.2(-0.10 - 0.056)(0.09 - 0.08) + 0.2(0 - 0.056)(0.22 - 0.08) + 0.2(0.10 - 0.056)(-0.06 - 0.08) \\
&\quad + 0.2(0.20 - 0.056)(-0.10 - 0.08) + 0.2(0.08 - 0.056)(0.25 - 0.08) \\
&= -0.00748
\end{aligned}$$

To see how strongly correlated these two assets are, let's calculate the coefficient of correlation.

$$\rho_{1,3} = \frac{\sigma_{13}}{\sigma_1 \sigma_3} = -0.5236$$

Conclusion: Assets 1 and 3 are relatively strongly negatively correlated. Thus, we have:

$$\sigma^2(r_I) = 0.003826 \quad \Rightarrow \quad \sigma(r_I) = 0.0619$$

Question: What have we accomplished in combining assets 1 and 3 together?

We have seen that the expected return of the portfolio is equal to the weighted average of the expected return of each asset in the portfolio. Is this also true for the risk (standard deviation)? Let's verify.

The weighted average of the risk of each asset in the portfolio is:

$$w_1\sigma(r_1) + w_3\sigma(r_3) = 0.12128$$

Thus, we have:

$$\sigma(r_I) < w_1\sigma(r_1) + w_3\sigma(r_3)$$

Conclusion: The risk of portfolio I is less than the weighted average of the risk of assets 1 and 3. This corresponds to the **diversification effect**.

Question: Is this always true or does it depend on the fact that assets 1 and 3 are negatively correlated? Consider a second portfolio.

Portfolio II: $(w_1, w_2) = (0.5, 0.5)$

$$\begin{aligned}
E(r_{II}) &= w_1E(r_1) + w_2E(r_2) \\
&= 0.5(0.056) + 0.5(0.058) \\
&= 0.057
\end{aligned}$$

and

$$\sigma^2(r_{II}) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \sigma_{12}$$

with

$$\sigma_{12} = 0.00379$$

and

$$\rho_{1,2} = 0.492$$

Portfolio II contains two assets that are positively correlated. Thus, we have:

$$\sigma^2(r_{II}) = 0.0059 \quad \Rightarrow \quad \sigma(r_{II}) = 0.0768$$

Let's compare this level of risk with the weighted average of the risk of assets 1 and 2.

$$w_1\sigma(r_1) + w_2\sigma(r_2) = 0.0886$$

Thus, we have:

$$\sigma(r_{II}) < w_1\sigma(r_1) + w_2\sigma(r_2)$$

Conclusion: The risk of portfolio II is also less than the weighted average of the risk of assets 1 and 2.

Remarks:

- In both cases, the risk of the portfolio is less than the weighted average of the risk of assets in the portfolio.
- The reduction in risk (diversification effect) is larger for Portfolio I than for Portfolio II.
- This is explained by the magnitude of the correlation between the two assets.
- The less correlated the assets are, the larger the diversification effect.
- The more correlated the assets are, the smaller the diversification effect.
- The diversification effect exists even if the two assets have a zero correlation.

5.5 Diversification with N Risky Assets

Consider a portfolio of N risky assets. The number of variances and covariances depend on the number of assets in the portfolio.

Nbr. assets in the portfolio	Nbr. of terms in the equation	Nbr. variances	Nbr. covariances
1	1	1	0
2	4	2	2
3	9	3	6
10	100	10	90
100	10 000	100	9 900
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	N^2	N	$N(N-1)$

As we have seen, the variance of a portfolio with $N = 2$ is given by:

$$\sigma^2(r_p) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \sigma_{12}$$

This equation has 4 terms:

- 2 variances: σ_1^2, σ_2^2
- 2 covariances: $2\sigma_{12}$

The variance of a portfolio with $N = 3$ is given by:

$$\sigma^2(r_p) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2w_1w_2 \sigma_{12} + 2w_1w_3 \sigma_{13} + 2w_2w_3 \sigma_{23}$$

This equation has 9 terms:

- 3 variances: $\sigma_1^2, \sigma_2^2, \sigma_3^2$
- 6 covariances: $2\sigma_{12}, 2\sigma_{13}, 2\sigma_{23}$

A portfolio with N assets will then have N^2 terms with N variances and $N^2 - N = N(N - 1)$ covariances. This makes it difficult to write the variance equation in an explicit way.

An alternative and more compact way for writing the variance of a portfolio is the following:

$$\sigma^2(r_p) = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$$

One can verify this by using 2 assets. If $N=2$, we have 4 possible combinations in the above expression:

- $i = 1, j = 1$: $w_1 w_1 \sigma_{11} = w_1^2 \sigma_1^2$
- $i = 1, j = 2$: $w_1 w_2 \sigma_{12}$
- $i = 2, j = 1$: $w_2 w_1 \sigma_{21}$
- $i = 2, j = 2$: $w_2 w_2 \sigma_{22} = w_2^2 \sigma_2^2$

Therefore, $\sigma^2(r_p) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \sigma_{12}$.

Note: One should note that the variance is simply a covariance of an asset with itself (i.e. $\sigma_{11} = \sigma_1^2$) and that covariances between two assets are symmetric $\sigma_{ij} = \sigma_{ji}$.

Proposition: The risk (as measured by the variance) of a diversified portfolio tends towards the average covariance of the assets in the portfolio.

Let $w_i = w_j = 1/N$. The variance equation then becomes:

$$\sigma^2(r_p) = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij}$$

This equation contains terms in variance ($i = j$) and terms in covariance ($i \neq j$). Let's separate these terms.

$$\sigma^2(r_p) = \frac{1}{N^2} \sum_{i=1}^N \sigma_i^2 + \frac{1}{N^2} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \sigma_{ij}$$

Consider the first term which captures the individual variances.

$$\frac{1}{N^2} \sum_{i=1}^N \sigma_i^2 = \frac{1}{N} \sum_{i=1}^N \frac{\sigma_i^2}{N} = \frac{1}{N} \bar{\sigma}^2 \quad \text{since} \quad \sum_{i=1}^N \frac{\sigma_i^2}{N} = \bar{\sigma}^2$$

Consider the second term which captures the covariances.

$$\frac{1}{N^2} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \sigma_{ij} = \frac{(N-1)}{N} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\sigma_{ij}}{N(N-1)} = \left(1 - \frac{1}{N}\right) \bar{\sigma}_{ij} \quad \text{since} \quad \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\sigma_{ij}}{N(N-1)} = \bar{\sigma}_{ij}$$

Thus, we can write,

$$\sigma^2(r_p) = \frac{1}{N} \bar{\sigma}^2 + \left(1 - \frac{1}{N}\right) \bar{\sigma}_{ij}$$

Question: What is the impact of increasing the number of assets on the variance of a portfolio?

Let's first examine the individual variances component. If we let N go to a sufficiently large number ($N \rightarrow \infty$), we get

$$\lim_{N \rightarrow \infty} \frac{1}{N} \bar{\sigma}^2 = 0$$

Thus, by simply increasing the number of assets to a sufficiently large number, we are able to eliminate the impact of individual variances on the risk of a diversified portfolio.

Is this also true for the covariance? Let's examine the second term of the variance equation.

If we let N go to a sufficiently large number ($N \rightarrow \infty$), we get:

$$\lim_{N \rightarrow \infty} \left(1 - \frac{1}{N}\right) \bar{\sigma}_{ij} = \bar{\sigma}_{ij} \neq 0$$

The implication of the above result is there there are two types of risk:

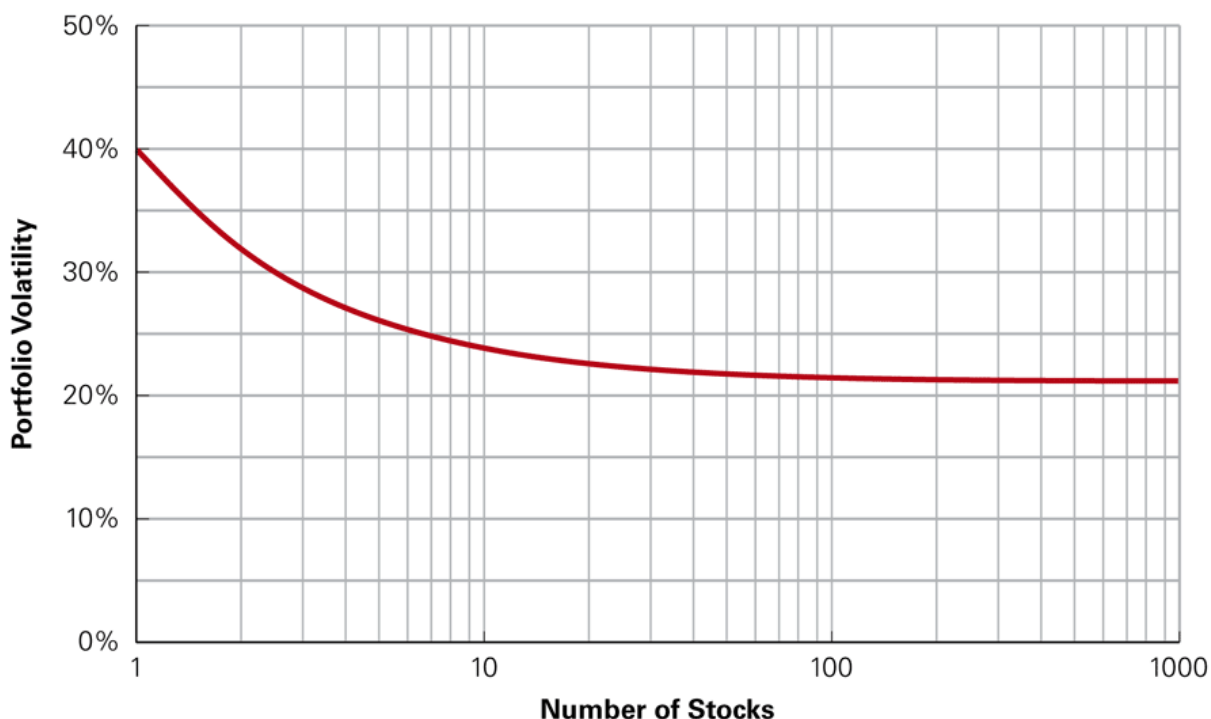
1. **Non-systematic**: this risk is **specific** to the asset or the firm (management, industry, size, ...)
2. **Systematic**: this risk is commonly called the **market** risk (variation in GDP, interest rates, inflation,...)

Diversification can help you reduce or eliminate non-systematic risk but you will not be able to eliminate systematic risk. Yet, it can be managed (see § CAPM).

The number of assets necessary to eliminate most non-systematic risk: 25 - 30 assets.

This is depicted in the following graph.

Figure 11.2 Volatility of an Equally Weighted Portfolio Versus the Number of Stocks



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5.6 Short Selling

An investor can hold a negative proportion of an asset in his portfolio, that is $w_i < 0$. This is called **short selling**.

Example: Suppose that on Feb 15, an investor wants to sell 500 shares of Total but he does not own the shares.

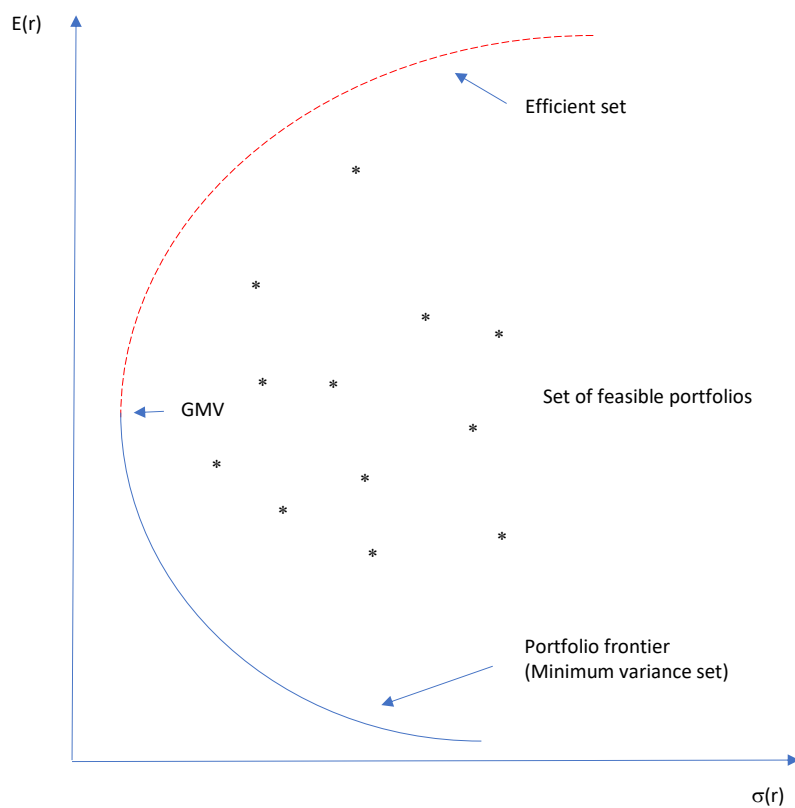
- Call his broker and ask him to sell 500 shares of Total.
- His broker borrows the shares from another client and sells them on the market.
- The investor can hold his position for as long as he wants as long as the other client does not need his shares.
- The market price of one share is 42 €.
- Revenue from selling the shares: $500 \times 42 = 21\,000$ €.
- Investor buys back the shares on March 15 at 37 €.
- Cost of the transaction: $500 \times 37 = 18\,500$ €
- Profit from this transaction: $21\,000 - 18\,500 = 2\,500$ € (before transaction costs)

Suppose that Total pays a dividend of 1 € on March 1. Then, the investors must compensate the client for the dividend he should have received for his 500 shares. The cost of this is 500 € which means that the profit of the strategy is now 2 000 €.

Remark: Short selling is a risky activity. Indeed, the price of the asset sold short can increase and your strategy could results in negative profits. Short selling is interesting is you expect a decrease in the price of a stock or a lower increase in the price of a stock relative to the price of other stocks.

5.7 Portfolio Frontier with Risky Assets

Consider an economy with N assets. By combining assets, we can construct different portfolios.



Def: The **portfolio frontier** or **minimum variance set** is the set of all portfolios, among the set of all feasible portfolios, offering the lowest risk (standard deviation) for a given expected return. This is a local minimum and the solution to the following problem:

$$\min_{w_i} \sigma(r_p) \quad \ni \quad E(r_p) = k$$

Def: The **mean - variance efficient set** is the set of all portfolios, among all minimum variance portfolios, offering the highest expected return for a given level of risk. It corresponds to the upper part of the portfolio frontier. It is the solution to:

$$\max_{w_i} E(r_p) \quad \ni \quad \sigma(r_p) = k$$

Note: The shape of the portfolio frontier is explained by the fact that equation of the frontier is a quadratic equation in the space $[E(r_p), \sigma^2(r_p)]$.

$$\sigma^2(r_p) = a + bE(r_p) + cE(r_p)^2$$

and an hyperbola in the space $[E(r_p), \sigma(r_p)]$

$$\sigma(r_p) = \sqrt{a E(r_p)^2 + b E(r_p) + c}$$

Def: The **global minimum variance portfolio (GMV)** is the portfolio, among all minimum variance portfolios, offering the lowest variance. This is a global minimum. It is the solution to:

$$\min_{w_i} \sigma^2(r_p)$$

In the case of two assets, the solution is given by:

$$\frac{\partial \sigma^2(r_p)}{\partial w_1} = 2w_1 \sigma_1^2 - 2(1 - w_1) \sigma_2^2 + 2 \sigma_1 \sigma_2 \rho_{1,2} - 4w_1 \sigma_1 \sigma_2 \rho_{1,2} = 0$$

Hence,

$$w_1^g = \frac{\sigma_2^2 - \sigma_1 \sigma_2 \rho_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_1 \sigma_2 \rho_{1,2}} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_{12}}$$

Construction of a Portfolio Frontier: Example with 2 assets

Construction of a Portfolio Frontier with 2 Assets

State of nature	Probability	Return Asset 1	Return Asset 2	Return Asset 3
A	0,3	0,05	0,30	-0,25
B	0,5	0,20	0,15	0,15
C	0,2	-0,15	-0,10	0,23

Expect. Return	0,0850	0,1450	0,0460
Variance	0,0180	0,0192	0,0385
Standard Deviation	0,1343	0,1387	0,1961

Cov(1,2)	0,0102
Corr(1,2)	0,5466

Cov(1,3)	0,0004
Corr(1,3)	0,0167

Cov(2,3)	-0,0225
Corr(2,3)	-0,8281

Portfolio 1: Assets 1 et 2

W1	W2	E(r)	Stand. Dev.	Variance
1,5	-0,5	0,0550	0,1735	0,0301
1,25	-0,25	0,0700	0,1517	0,0230
1	0	0,0850	0,1343	0,0180
0,9	0,1	0,0910	0,1289	0,0166
0,8	0,2	0,0970	0,1247	0,0156
0,7	0,3	0,1030	0,1218	0,0148
0,6	0,4	0,1090	0,1202	0,0144
0,5	0,5	0,1150	0,1200	0,0144
0,4	0,6	0,1210	0,1212	0,0147
0,3	0,7	0,1270	0,1238	0,0153
0,2	0,8	0,1330	0,1276	0,0163
0,1	0,9	0,1390	0,1326	0,0176
0	1	0,1450	0,1387	0,0192
-0,25	1,25	0,1600	0,1575	0,0248
-0,5	1,5	0,1750	0,1803	0,0325

Global Min. variance portfolio

W1*	0,5355
W2*	0,4645

E(Return)	0,1129
Variance	0,0144
Standrad Deviation	0,1199

Construction of a Portfolio Frontier with 2 Assets

Portfolio 2: Assets 2 et 3

W2	W3	E(r)	Stand. Dev.	Variance
1,5	-0,5	0,1945	0,2944	0,0867
1,25	-0,25	0,1698	0,2157	0,0465
1	0	0,1450	0,1387	0,0192
0,9	0,1	0,1351	0,1091	0,0119
0,8	0,2	0,1252	0,0815	0,0066
0,7	0,3	0,1153	0,0585	0,0034
0,6	0,4	0,1054	0,0476	0,0023
0,5	0,5	0,0955	0,0562	0,0032
0,4	0,6	0,0856	0,0782	0,0061
0,3	0,7	0,0757	0,1054	0,0111
0,2	0,8	0,0658	0,1348	0,0182
0,1	0,9	0,0559	0,1652	0,0273
0	1	0,0460	0,1961	0,0385
-0,25	1,25	0,0213	0,2745	0,0754
-0,5	1,5	-0,0035	0,3537	0,1251

Global Min. variance portfolio

W2* 0,5936

W3* 0,4064

E(Return) 0,1048

Variance 0,0023

Standrad Deviation 0,0476

Portfolio 3: Assets 1 et 3

W1	W3	E(r)	Stand. Dev.	Variance
1,5	-0,5	0,1045	0,2225	0,0495
1,25	-0,25	0,0948	0,1740	0,0303
1	0	0,0850	0,1343	0,0180
0,9	0,1	0,0811	0,1227	0,0151
0,8	0,2	0,0772	0,1150	0,0132
0,7	0,3	0,0733	0,1117	0,0125
0,6	0,4	0,0694	0,1134	0,0129
0,5	0,5	0,0655	0,1198	0,0143
0,4	0,6	0,0616	0,1302	0,0169
0,3	0,7	0,0577	0,1437	0,0207
0,2	0,8	0,0538	0,1596	0,0255
0,1	0,9	0,0499	0,1772	0,0314
0	1	0,0460	0,1961	0,0385
-0,25	1,25	0,0363	0,2469	0,0610
-0,5	1,5	0,0265	0,3006	0,0904

Global Min. variance portfolio

W1* 0,6838

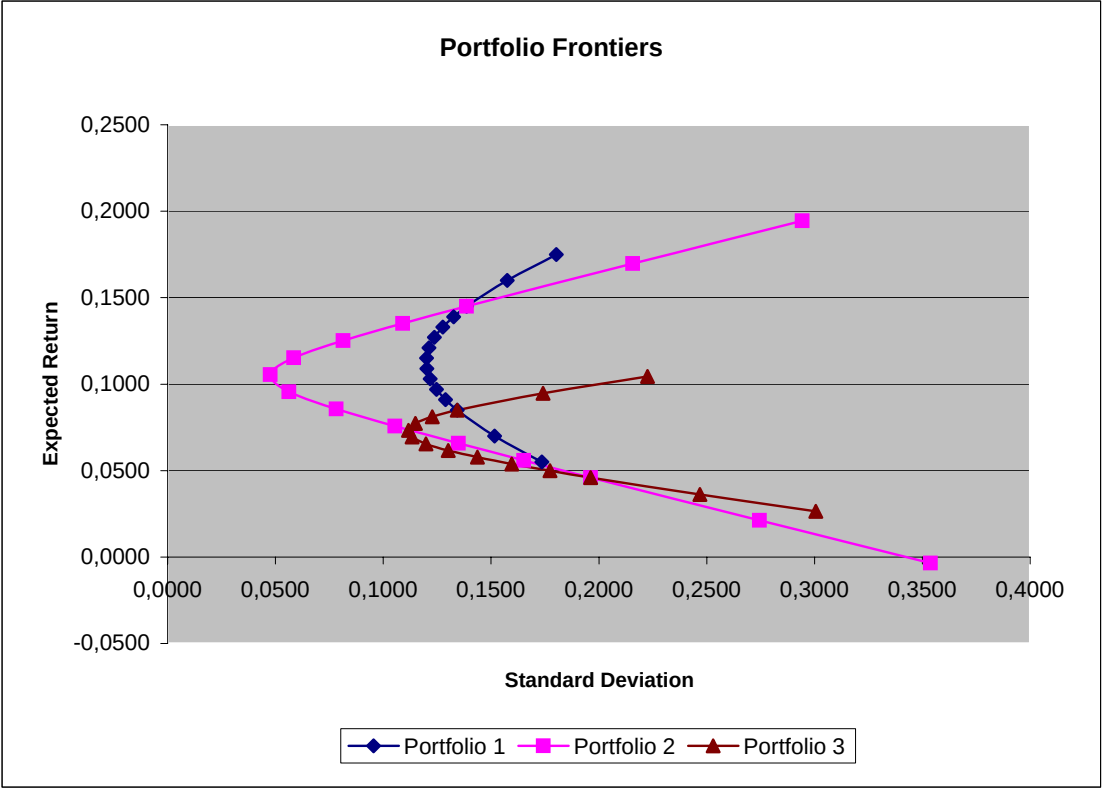
W3* 0,3162

E(Return) 0,0727

Variance 0,0125

Standrad Deviation 0,1116

Construction of a Portfolio Frontier with 2 Assets



5.8 Diversification: Two Special Cases

As we have seen, the coefficient of correlation between the return of two assets is bounded by $[-1, +1]$. Thus, there are two special cases in the context of portfolio analysis.

1. $\rho = +1$

Replacing $\rho = 1$ in the variance equation, we get:

$$\begin{aligned}\sigma^2(r_p) &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \sigma_1\sigma_2 \\ &= (w_1 \sigma_1 + w_2 \sigma_2)^2\end{aligned}$$

Therefore, we get that,

$$\sigma(r_p) = w_1 \sigma_1 + w_2 \sigma_2$$

If $\rho = 1$, the risk of a portfolio is equal to the weighted average of the risk of the individual assets in the portfolio $\Rightarrow$ **no diversification**

Solving for w_1 and replacing in $E(r_p)$, the equation of the portfolio frontier is a **straight line**.

The global minimum variance portfolio is given by:

$$w_1^g = \frac{\sigma_2}{\sigma_2 - \sigma_1} \quad \text{and} \quad w_2^g = 1 - w_1^g$$

2. $\rho = -1$

Replacing $\rho = -1$ in the variance equation, we get:

$$\begin{aligned}\sigma^2(r_p) &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 - 2w_1w_2 \sigma_1\sigma_2 \\ &= (w_1 \sigma_1 - w_2 \sigma_2)^2\end{aligned}$$

Therefore, we get that,

$$\sigma(r_p) = |w_1 \sigma_1 - w_2 \sigma_2|$$

Solving for w_1 and replacing in $E(r_p)$, the equation of the portfolio frontier is a **cone** with two straight line segments, one with a positive slope and one with a negative slope.

The global minimum variance portfolio is given by:

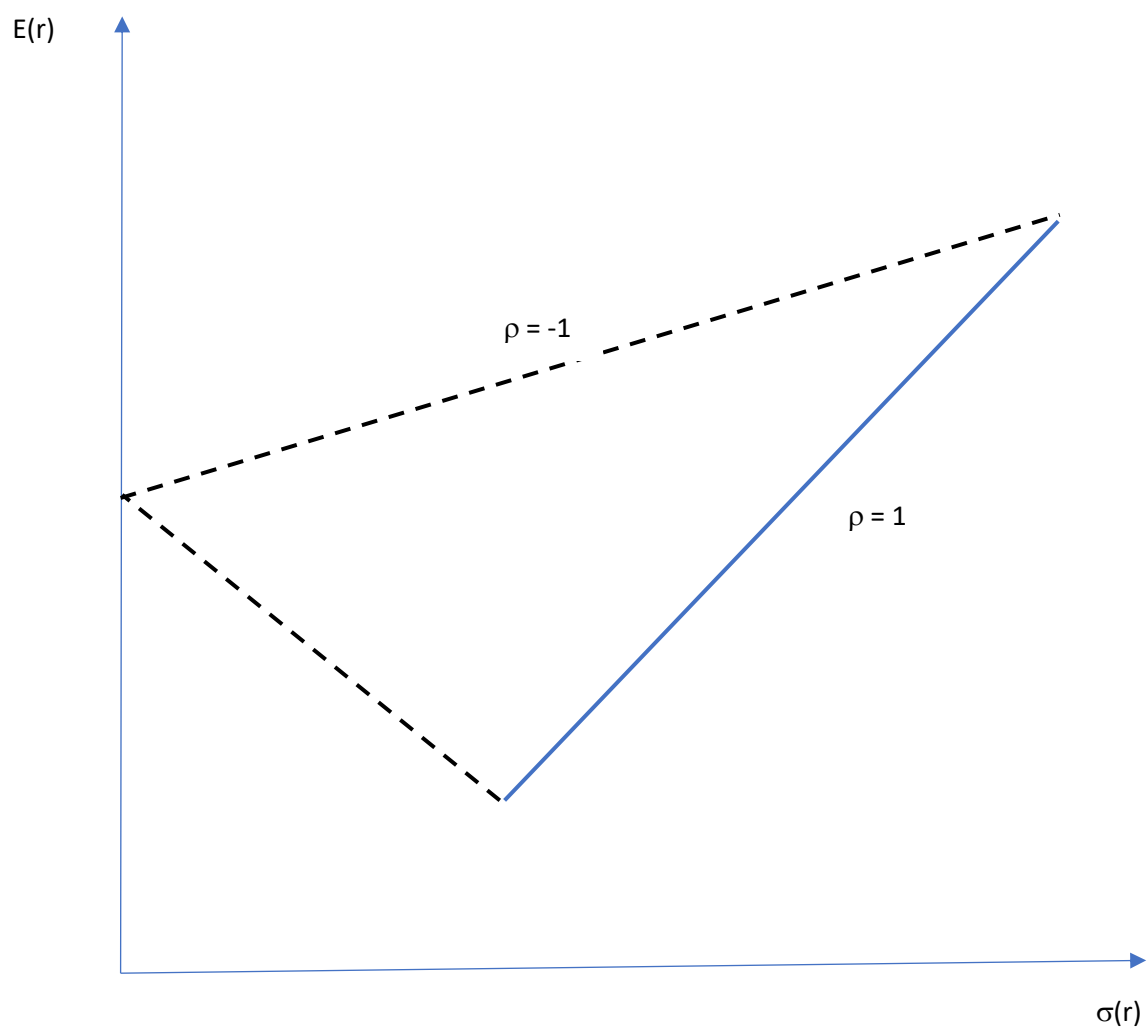
$$w_1^g = \frac{\sigma_2}{\sigma_1 + \sigma_2} \quad \text{and} \quad w_2^g = 1 - w_1^g$$

Note: The risk of a portfolio when $\rho = -1$ is less than its risk with $\rho = +1$.

We can illustrate this with the following example with the return of three assets (X, Y, Z) over the last five years.

Year	X	Y	Z	50%X + 50%Y	50%X + 50%Z
1	0.080	0.160	0.100	0.120	0.090
2	0.100	0.140	0.125	0.120	0.1125
3	0.120	0.120	0.150	0.120	0.135
4	0.140	0.100	0.175	0.120	0.1575
5	0.160	0.080	0.200	0.120	0.180
$E(r)$	0.120	0.120	0.150	0.120	0.135
$\sigma(r)$	0.0316	0.0316	0.0395	0.000	0.0356
$\rho_{x,y}$	-1				
$\rho_{x,z}$	+1				

1. Assets X and Y are perfectly negatively correlated ($\rho_{x,y} = -1$). Thus, the risk of the portfolio measured by the standard deviation is equal to: $\sigma_p = |w_x\sigma_x - w_y\sigma_y|$. Given that both assets have the same risk, an equally weighed portfolio of X and Y will have zero risk (global minimum variance portfolio).
2. Assets X and Z are perfectly positively correlated ($\rho_{x,z} = +1$). Thus, the risk of the portfolio is equal to: $\sigma_p = w_x\sigma_x + w_z\sigma_z$. Thus, there is no diversification effect since the risk of the portfolio is just equal to the weighted average risk of the assets in the portfolio.



Conclusions:

1. If $\rho = +1$: no diversification.
2. If $\rho = -1$: maximum diversification.
3. Overall, a smaller correlation between assets generate a larger diversification effect.

Portfolio Theory: Application 1 (In class)

Consider two securities with the following distribution of returns:

State	Prob.	Return 1	Return 2
A	0.10	0.00	0.60
B	0.80	0.25	0.20
C	0.10	0.50	-0.20

1. Calculate the expected return and the variance of both assets and the covariance and the coefficient of correlation between the two securities.
2. Plot the portfolio frontier based on the following portfolios:
 $(w_1, w_2) = (1 ; 0), (0,75 ; 0,25), (0,25 ; 0,75), (0 ; 1)$.
3. How do you explain that security 2 is mean-variance dominated by security 1 but yet enters all efficient portfolios but one.

5.9 Portfolio Frontier and Correlation

Let's examine how the portfolio frontier is affected by a change in the coefficient of correlation under two assumptions:

1. no short selling
2. short selling

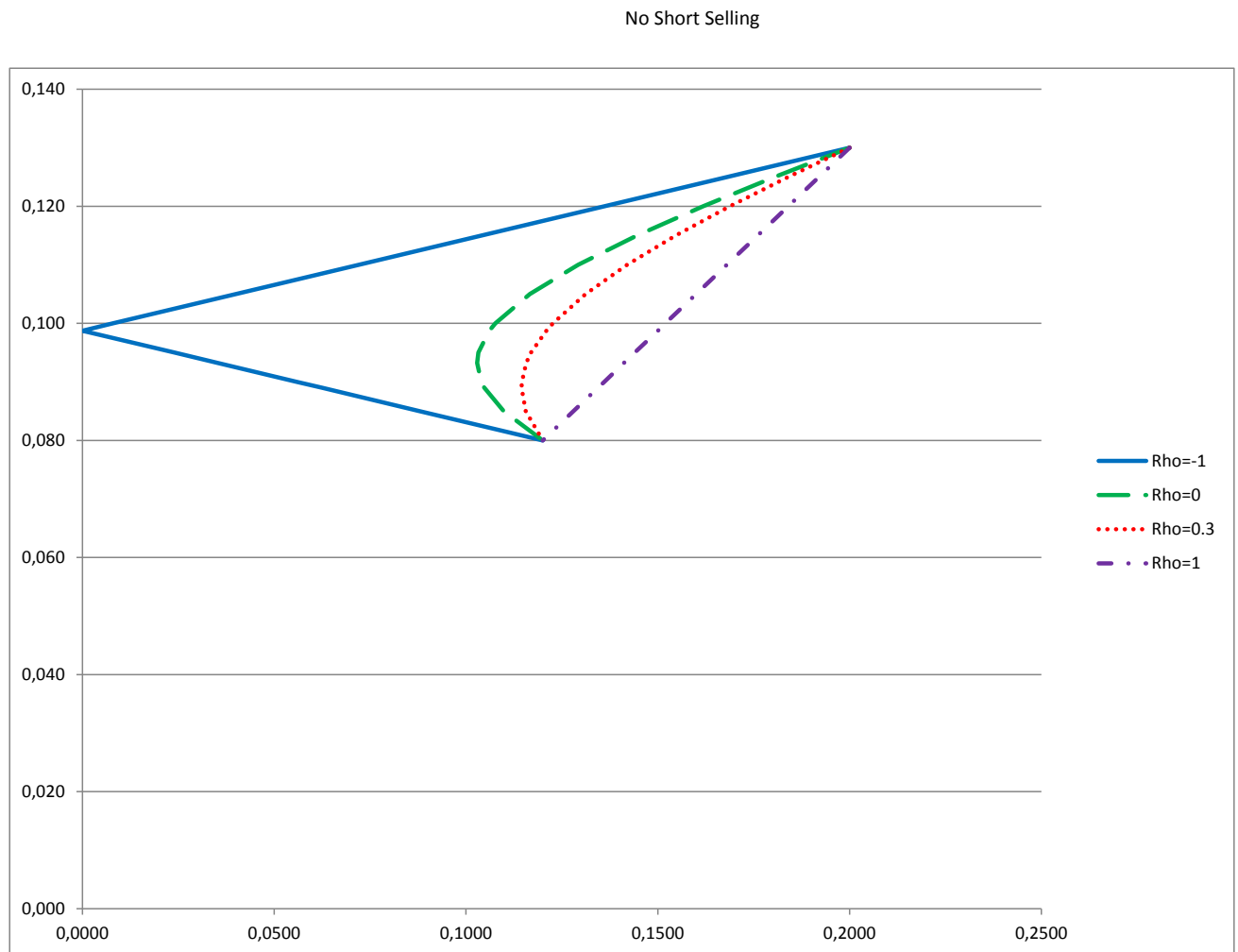
Scenario I: No short selling

No Short Selling

E(r1)	0,08
E(r2)	0,13
$\sigma(r1)$	0,12
$\sigma(r2)$	0,2

W1	W2	E(rp)	Standard Deviation			
			$\rho = -1$	$\rho = 0$	$\rho = 0,3$	$\rho = 1$
0	1	0,130	0,2000	0,2000	0,2000	0,2000
0,1	0,9	0,125	0,1680	0,1804	0,1840	0,1920
0,2	0,8	0,120	0,1360	0,1618	0,1688	0,1840
0,3	0,7	0,115	0,1040	0,1446	0,1547	0,1760
0,4	0,6	0,110	0,0720	0,1292	0,1420	0,1680
0,5	0,5	0,105	0,0400	0,1166	0,1311	0,1600
0,6	0,4	0,100	0,0080	0,1076	0,1226	0,1520
0,6250	0,3750	0,099	0,0000	0,1061	0,1209	0,1500
0,7	0,3	0,095	0,0240	0,1032	0,1170	0,1440
0,7353	0,2647	0,093	0,0353	0,1029	0,1157	0,1412
0,8	0,2	0,090	0,0560	0,1040	0,1145	0,1360
0,8200	0,18	0,089	0,0624	0,1048	0,1145	0,1344
0,9	0,1	0,085	0,0880	0,1098	0,1156	0,1280
1	0	0,080	0,1200	0,1200	0,1200	0,1200

Min Var Port						
W1			0,6250	0,7353	0,8200	2,5000
W2			0,3750	0,2647	0,1800	-1,5000
E(rp)			0,0988	0,0932	0,0890	0,0050
$\sigma(rp)$			0,0000	0,1029	0,1145	0,0000



Remarks:

1. The portfolio frontier when $\rho = +1$ (purple dotted line) is a straight line.
2. The portfolio frontier when $\rho = -1$ (blue line) is a cone with two straight line segments going through the vertical axis $\sigma_p = 0$
3. A reduction in the correlation between assets implies a larger curvature of the portfolio frontier $\Rightarrow$ more diversification effect.

Scenario II: Short selling

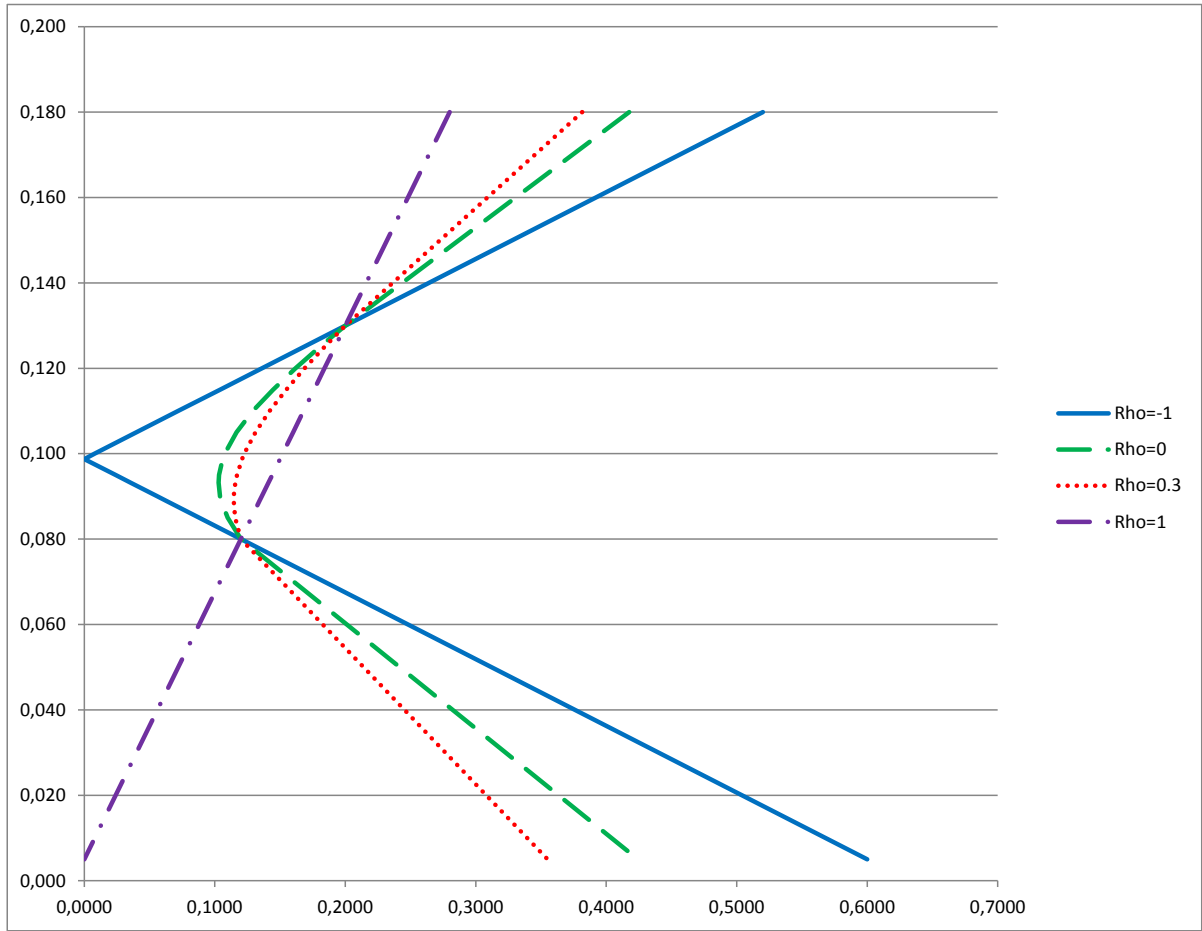
With Short Selling

E(r1)	0,08
E(r2)	0,13
$\sigma(r1)$	0,12
$\sigma(r2)$	0,2

W1	W2	E(rp)	Standard Deviation			
			$\rho = -1$	$\rho = 0$	$\rho = 0,3$	$\rho = 1$
-1	2	0,180	0,5200	0,4176	0,3816	0,2800
0	1	0,130	0,2000	0,2000	0,2000	0,2000
0,1	0,9	0,125	0,1680	0,1804	0,1840	0,1920
0,2	0,8	0,120	0,1360	0,1618	0,1688	0,1840
0,3	0,7	0,115	0,1040	0,1446	0,1547	0,1760
0,4	0,6	0,110	0,0720	0,1292	0,1420	0,1680
0,5	0,5	0,105	0,0400	0,1166	0,1311	0,1600
0,6	0,4	0,100	0,0080	0,1076	0,1226	0,1520
0,6250	0,3750	0,099	0,0000	0,1061	0,1209	0,1500
0,7	0,3	0,095	0,0240	0,1032	0,1170	0,1440
0,7353	0,2647	0,093	0,0353	0,1029	0,1157	0,1412
0,8	0,2	0,090	0,0560	0,1040	0,1145	0,1360
0,8200	0,18	0,089	0,0624	0,1048	0,1145	0,1344
0,9	0,1	0,085	0,0880	0,1098	0,1156	0,1280
1	0	0,080	0,1200	0,1200	0,1200	0,1200
2,5	-1,5	0,005	0,6000	0,4243	0,3550	0,0000

Min Var Port				
W1	0,6250	0,7353	0,8200	2,5000
W2	0,3750	0,2647	0,1800	-1,5000
E(rp)	0,0988	0,0932	0,0890	0,0050
$\sigma(rp)$	0,0000	0,1029	0,1145	0,0000

With Short Selling



Conclusions:

1. The introduction of short selling increased the number of possible portfolios and thus expanded the portfolio frontiers for all levels of correlation.
2. From two risky assets that are perfectly positively correlated ($\rho = +1$), we can construct a risk-free portfolio. The portfolio $(w_1, w_2) = (2.5, -1.5)$ has no risk as defined by the standard deviation.

Note: An increase in the number of assets considered to construct the frontier will lead to an expansion in the portfolio frontier and the efficient frontier. For instance, the frontier with two assets is within the frontier for 3 assets, etc.

5.10 Portfolio Frontier with Risky Assets and a Risk Free Asset

The existence of a risk-free asset means that investors have the possibility to borrow or lend at the risk free interest rate, r_f . The characteristics of this asset are:

$$\begin{aligned} E(r_f) &= r_f \\ \sigma(r_f) &= 0 \end{aligned}$$

Consider a portfolio made up of the risk free asset and a portfolio of risky assets (M). The characteristics of this portfolio are:

$$\begin{aligned} E(r_p) &= w_f r_f + w_m E(r_m) \\ &= (1 - w_m) r_f + w_m E(r_m) \\ &= r_f + w_m [E(r_m) - r_f] \end{aligned}$$

and

$$\begin{aligned} \sigma^2(r_p) &= w_f^2 \sigma_f^2 + w_m^2 \sigma_m^2 + 2w_f w_m \sigma_f \sigma_m \rho_{m,f} \\ &= w_m^2 \sigma_m^2 \\ \sigma(r_p) &= |w_m \sigma_m| \end{aligned}$$

We can construct the efficient portfolio frontier using these two equations. Indeed:

$$w_m = \frac{\sigma_p}{\sigma_m}$$

Replacing in the expected return equation:

$$\begin{aligned} E(r_p) &= r_f + w_m [E(r_m) - r_f] \\ &= r_f + \left[\frac{E(r_m) - r_f}{\sigma_m} \right] \sigma_p \end{aligned}$$

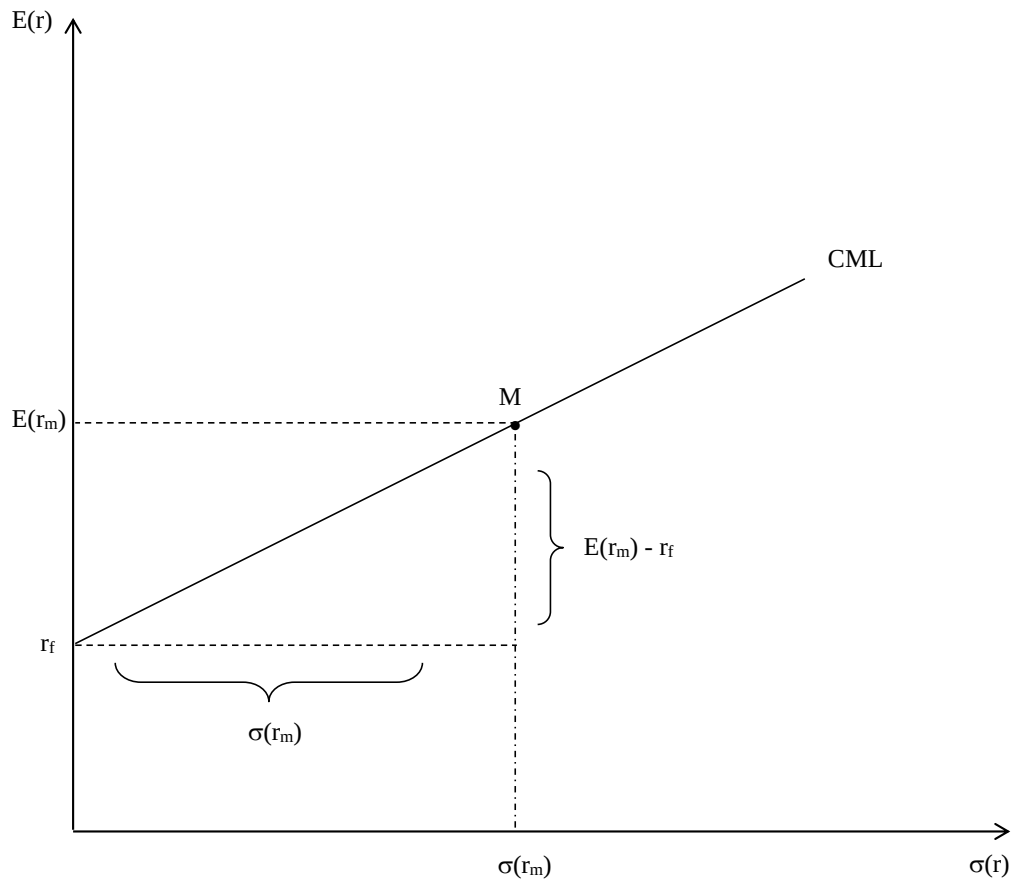
Conclusion: In an economy with one risk free asset and N risky assets, the portfolio frontier is a **straight line**. This line is called the **Capital Market Line (CML)** or the **Markowitz line**.

The slope of the market line is called the **Sharpe** ratio:

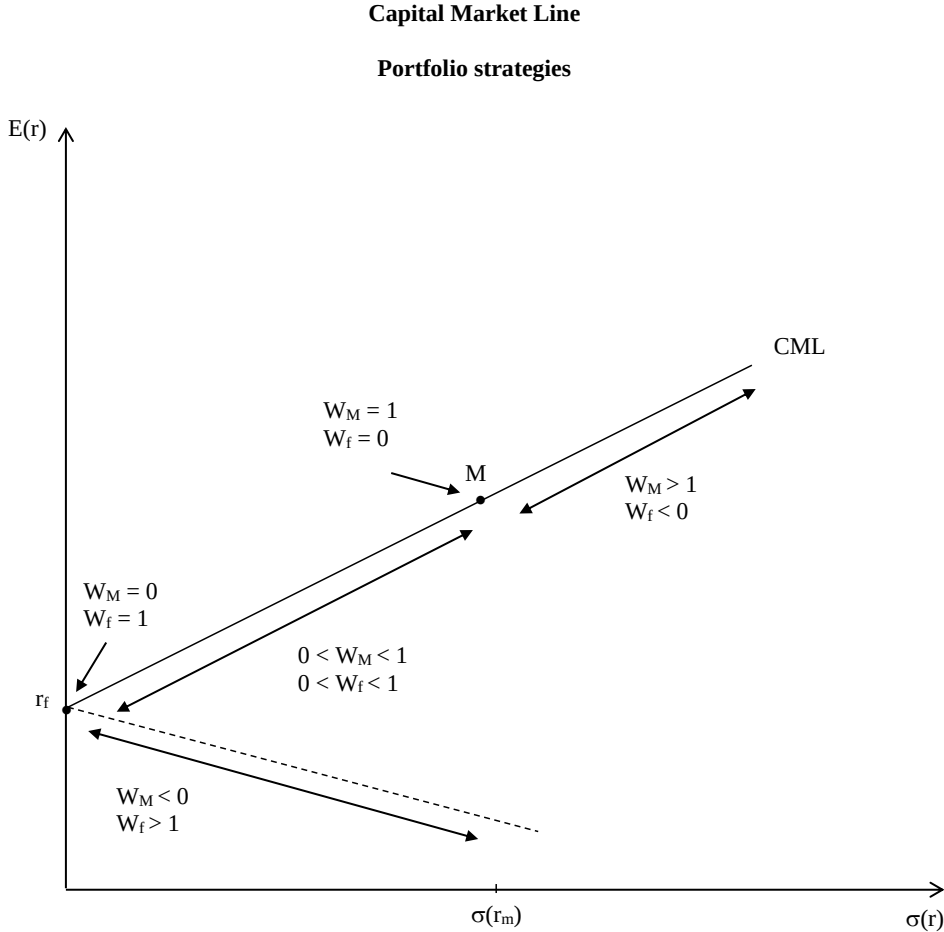
$$\text{Sharpe ratio} = \frac{E(r_m) - r_f}{\sigma_m}$$

This ratio measures the return-volatility ratio of an asset or a portfolio. It measures the excess return of an asset per unit of risk.

Capital Market Line



From the CML, we can identify all possible combinations of risky and risk-free assets.



There are a number of strategies that an investor can adopt.

- Portfolio ($w_M = 0, w_f = 1$): Invest 100% in the risk-free asset (Treasury Bills) and nothing in the risky assets. The return is equal to the risk-free rate and there is no risk.
- Portfolio ($w_M = 1, w_f = 0$): Invest 100% in the risky assets M and nothing in the risk-free asset. The expected return and the risk of the portfolio correspond to the one of M .
- ($0 < w_M < 1, 0 < w_f < 1$): Invest a positive proportion of the wealth in both assets. The investor holds the risk-free asset and some risky assets. Holding the risk-free asset is equivalent to **lending** (buying Treasury Bills).
- ($w_M > 1, w_f < 0$): Invest more than 100% of the wealth in M and a negative proportion of your wealth (short selling) in the risk-free asset. Short selling the risk-free asset is equivalent to **borrowing**.
- ($w_M < 0, w_f > 1$): Invest more than 100% of the wealth in the risk free-asset and a negative proportion of your wealth (short selling) in the risky assets. One can see that these strategies will never be implemented since they are dominated by strategies on the efficient frontier.

Portfolio Theory: Application 2 (In class)

Consider the following information on a portfolio of risky assets M and the risk-free asset.

- $r_f = 0.04$
- $E(r_m) = 0.10$
- $\sigma(r_m) = 0.07$

1. Complete the Table.

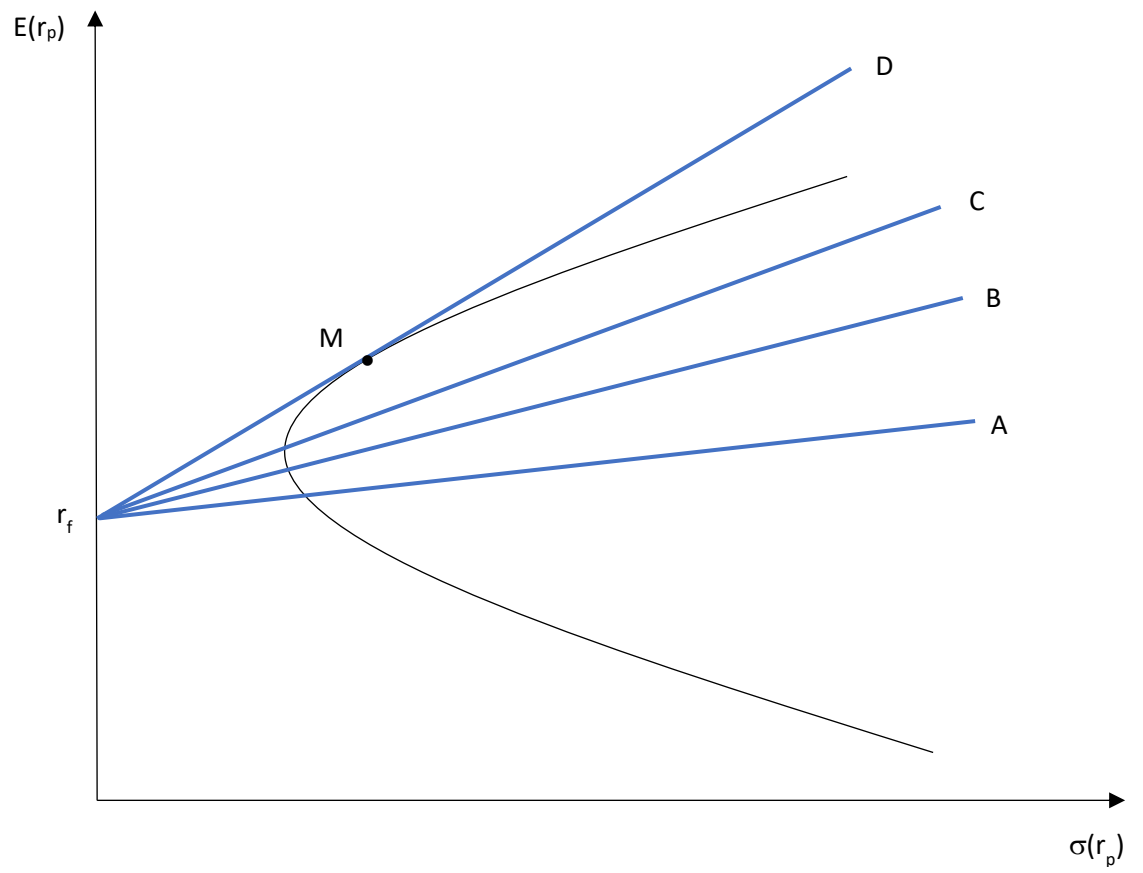
w_m	w_f	$E(r_p)$	$\sigma(r_p)$
-0.50	1.50		
-0.25	1.25		
0	1		
0.25	0.75		
0.50	0.50		
0.75	0.25		
1	0		
1.25	-0.25		
1.50	-0.50		

2. Plot the Capital Market Line (CML).
3. Identify the inefficient portfolios. Why are they inefficient?
4. What is the Sharpe ratio?
5. Assuming that you have 10 000€ to invest and that you aim for an expected return of 8.5%, what should be the composition of your portfolio?
6. What about if you expected return objective is 12%. What would that portfolio be and what would be its risk?

As we have seen, the CML is composed of portfolios combining the risk-free asset and risky assets.

Question: What criteria do we use to select these risky assets?

Answer: Consider the following graph.



Choice of the optimal portfolio: A step by step analysis

- In an economy with a risk-free asset, all efficient portfolios are on a straight line with a positive slope with an intercept equal to r_f .
- Let's consider segment A . Can the portfolio of risky assets be on this segment? The answer is NO since they are dominated by portfolios on segment B .
- Will investors select portfolios on segment B ? The answer again is NO since they are dominated by portfolios on segments above it.
- This selection process continues until we reach the limit of the feasible set with risky assets only, that is segment C . We cannot go beyond this segment since these portfolios are not in the feasible set, that is there are no combinations of risky assets that would generate these portfolios.
- Thus, the optimal portfolio to combine with the risk-free asset is represented by point M which is the portfolio tangent to the CML and the portfolio frontier with risky assets.
- M is the portfolio that maximizes the slope of the CML, the Sharpe ratio. The solution to this maximization problem is called the **tangent portfolio**. It is the solution to:

$$\max_{w_i} \frac{E(r_m) - r_f}{\sigma_m} \quad \ni \quad \sum_{i=1}^n w_i = 1$$

Theorem: In an economy with a risky assets and a risk-free asset, all efficient portfolios are a combination of the risk-free asset and the tangent portfolio which contains risky assets only.

This theorem is commonly called the **Two-Fund Separation Theorem** and says that all investors will hold only two assets in their portfolio:

- risk-free asset (borrow or lend)
- tangent portfolio

Thus, the portfolio decision is a 2-step process:

- Identification of M
- Combination of M with the risk-free asset (w_f, w_m) as a function of the level of risk than the investor is willing to support.

5.11 Portfolio M

One important question which remains to be answered is: What is M ?

To answer this question, one needs to go back to the assumption of the Mean-Variance model.

- No transaction costs
- Homogeneous expectations of investors on the return and risk of the assets

- All investors face the same risk-free rate

As we have seen, M is the portfolio which maximizes the Sharpe ratio. It is the solution to the following problem:

$$\max_{w_i} S = \frac{E(r_m) - r_f}{\sigma_m}$$

- All investors are then facing the same mathematical problem and there exist only one solution to it.
- M will be the same for all investors.
- M is the **market portfolio** which includes all the risky assets in the economy.
- Each asset i in the market portfolio is represented in proportion to its market value, that is:

$$w_i = \frac{V_i}{\sum V_i}$$

where V_i is the market value of asset i and $\sum V_i$ is the market value of M .

Intuitive explanation of why M has all the assets in the economy.

- Suppose that there are 100 risky assets in the economy.
- Each investor is asked to create a portfolio M of 100 assets $(w_1, w_2, w_3, \dots, w_{100})$ which maximizes the Sharpe ratio.
- We know from above that the solution to this problem is unique. M is the same for all investors and thus each investor needs to create the exact same portfolio.
- Without coordination, the likelihood that all investors choose the same portfolio is ZERO.
- Indeed, there is an infinite number of combinations $(w_1, w_2, w_3, \dots, w_{100})$ of 100 assets.
- However, if all investors agree on two rules:
 - Select all the assets
 - Weight each asset in proportion of the market value, $V_i / \sum V_i$
- This portfolio will be the same for everyone and this is the market portfolio.

This raises two important questions: Is this result:

1. Reasonable?

The selection of all the assets in the economy derives from the assumption of the absence of transaction costs and the gains from diversification. We have seen that diversification is a matter of the number of assets. By simply increasing the number of assets in a portfolio, an investor can reduce and almost eliminate the non-systematic risk.

Thus, in the absence of transaction costs, the only logical solution is to take all the assets in the economy.

2. Feasible?

In the real world, transaction costs are positive. Thus, the limit on the number of assets in a portfolio is reached when the marginal cost of adding assets (transaction costs) is equal to the marginal benefit in terms of risk reduction.

In addition, the market portfolio with all the assets in the economy does not exist. An investor cannot buy such a portfolio, although some large index aim at capturing such a portfolio. Stock market index (i.e. S&P 500) have been used to proxy the market portfolio.

6 Capital Asset Pricing Model (CAPM)

Starting point of the CAPM:

- According to the Two-Fund Separation Theorem, all investors hold two funds: i) the risk-free asset and ii) the market portfolio
- In equilibrium, all assets are held by the investors and the price of each is such Supply = Demand. Assets that are not held have a zero price.
- The market portfolio is composed of all assets on a market value weight basis.

Assumptions:

1. 1 period
2. Perfect and competitive capital markets
3. Risk averse investors hold efficient portfolios of assets to maximize their expected utility in terms of expected return and risk
4. Homogeneous expectations of agents on the expected return and the variance in the return of assets
5. N risky assets
6. Agents can borrow or lend at the risk free asset and can short sell risky assets

6.1 Derivation of the CAPM

The idea of the CAPM is the following.

In equilibrium, the expected return of a risky asset is equal to the risk-free rate plus a risk premium which is proportional to the level of systematic (market) risk of the asset.

First, let's examine the risk of an asset in equilibrium. Consider the following return on JOJO and the market in four possible states of the economy.

State	Market	JOJO
High	15%	25%
High	15%	15%
Low	-5%	-5%
Low	-5%	-15%

Let's calculate the expected return in each state (High or Low)

State	Market	JOJO
High	15%	20%
Low	-5%	-10%

- The return of JOJO varies with the state of the economy and respond to changes in the market return.
- $\Delta \text{ Market return (High - Low)} = 15\% - (-5\%) = 20\%$
- $\Delta \text{ JOJO return (High - Low)} = 20\% - (-10\%) = 30\%$

•

$$\frac{\Delta \text{ JOJO return}}{\Delta \text{ Market return}} = \frac{30\%}{20\%} = 1.5$$

- Thus, if the market return varies by 1%, JOJO's return will vary by 1.5%.
- The responsiveness (sensitivity) coefficient of JOJO to the market is equal to 1.5.
- This coefficient is called the **Beta** of an asset.
- We can show that,

$$\beta_i = \frac{Cov(r_i, r_m)}{\sigma_m^2} = \frac{\sigma_{im}}{\sigma_m^2}$$

- This is a measure of **systematic (market) risk**

Therefore, in equilibrium, we can write:

$$E(\tilde{r}_i) = r_f + \beta_i [E(\tilde{r}_m) - r_f] \quad \forall i$$

In equilibrium, the expected return of an asset has two components.

- Risk-free rate: r_f
- Risk premium: $\beta_i [E(\tilde{r}_m) - r_f]$

In addition, the risk premium has two components:

- Market or equity premium: $E(\tilde{r}_m) - r_f$
- Systematic risk: β_i

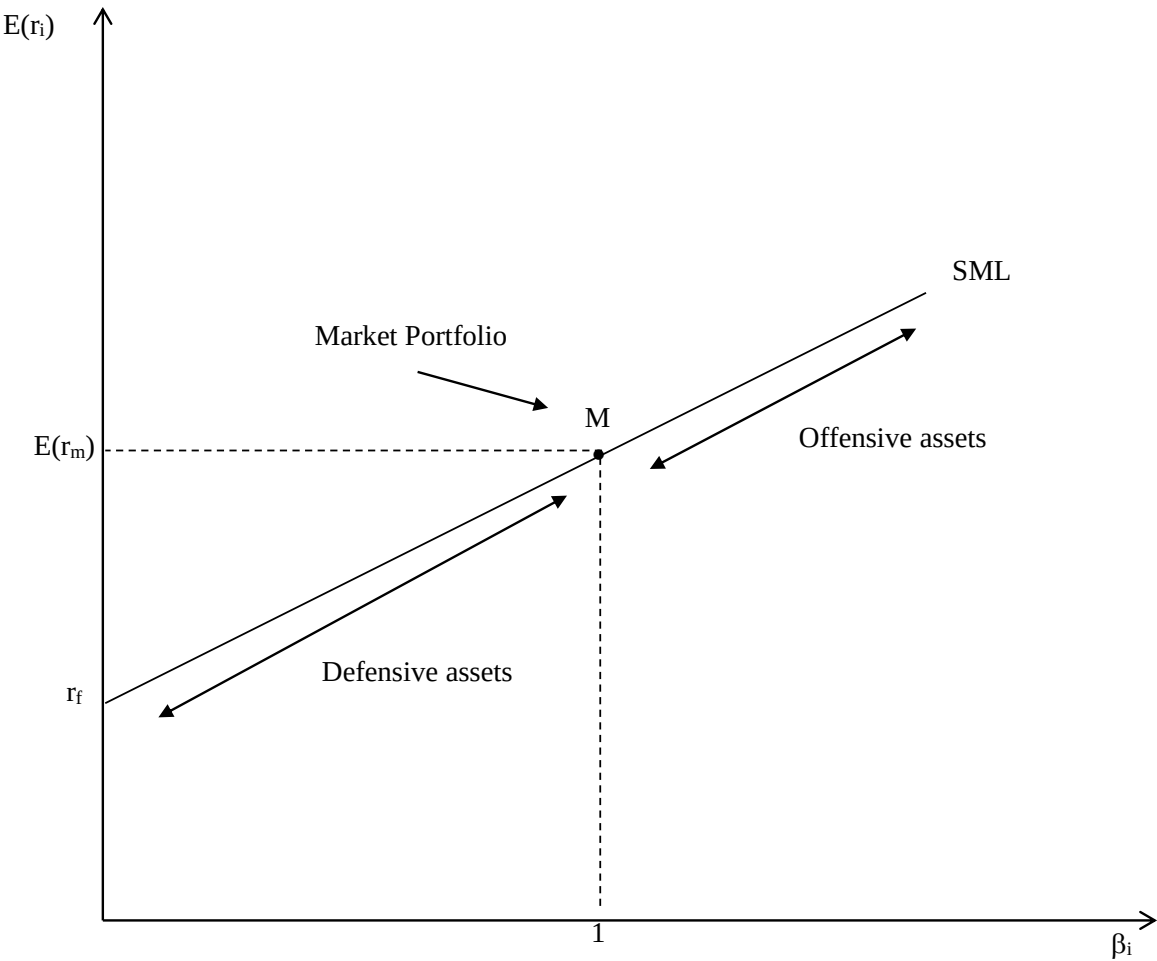
The main message from the CAPM is that the expected return of an asset depends only on its level of systematic risk as measured by its beta.

The CAPM equation is a straight line in the space $[E(\tilde{r}), \beta]$ with:

- positive intercept: r_f
- positive slope: $E(\tilde{r}_m) - r_f$

This line is called the **Security Market Line (SML)** and prevails for **all** assets or portfolios in the economy.

Securities Market Line



- $\beta_f = 0$
- $\beta_m = 1$
- $\beta > 1$: Offensive assets
- $\beta < 1$: Defensive assets

5Y Beta CAC 40**July 2026**

Firm	Beta	Firm	Beta
Accor	0,87	Legrand	1,00
Air Liquide	0,65	L'Oréal	0,89
Airbus Group	0,88	LVMH	0,84
Arcelormittal	1,74	Michelin	1,02
Axa	0,59	Orange	0,25
BNP Paribas	1,04	Pernod Ricard	0,47
Bouygues	0,69	Publicis	0,59
Bureau Veritas	0,74	Renault	0,81
Capgemini	0,65	Safran	0,96
Carrefour	0,62	Saint-Gobain	1,22
Crédit Agricole	0,81	Sanofi	0,28
Danone	0,19	Schneider Electric	1,16
Dassault Systèmes	0,53	Société Générale	0,98
Eiffage	0,72	Stellantis (Peugeot)	0,99
ENGIE	0,57	STMicroelectronics	1,56
Essilor Int'l	0,55	Thales	0,11
Eurofins Scientific	0,86	Total Energies	0,05
Euronext	0,85	Unibail-Rodamco-Westfield	1,40
Hermes Int'l	1,00	Véolia Environnement	1,01
Kering (PPR)	0,98	Vinci	0,75

Source: Capital IQ

The β of an asset can be interpreted as the coefficient of a linear regression of the return of an asset on the return of the market portfolio.

According to the **Market Model**, there exists a statistical relationship between the return of any asset and the market return. Thus, we can write:

$$\tilde{r}_i = \alpha_i + \beta_i \tilde{r}_m + \tilde{\epsilon}_i$$

with $E(\tilde{\epsilon}_i) = 0$, $\text{Var}(\tilde{\epsilon}_i) = \sigma_{\tilde{\epsilon}_i}^2$, $E(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0$ for $i \neq j$ et $E(\tilde{\epsilon}_i, \tilde{r}_m) = 0$.

As we can see, the β of an asset measures the sensitivity of the asset to variations in the return of the market portfolio $\Rightarrow$ **systematic** or **market** risk. Indeed,

$$\frac{\partial \tilde{r}_i}{\partial \tilde{r}_m} = \beta_i$$

Using this equation, we can write:

$$E(\tilde{r}_i) = \alpha_i + \beta_i E(\tilde{r}_m)$$

and

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\tilde{\epsilon}_i}^2$$

Conclusion: In equilibrium, the risk of any asset measured by its variance has two components:

1. systematic variance (non diversifiable): $\beta_i^2 \sigma_m^2$
2. specific variance (diversifiable): $\sigma_{\tilde{\epsilon}_i}^2$

Example: Suppose two assets with the following characteristics:

- $\sigma_1^2 = \sigma_2^2 = 0.1$
- $\sigma_m^2 = 0.05$
- $\beta_1 = 1.2, \beta_2 = 0.8$

We can then decompose the variance of each asset in the following way:

- $\sigma_1^2 = 0.1$

$$\beta_1^2 \sigma_m^2 = 0.072$$

$$\sigma_{\epsilon_1}^2 = \sigma_1^2 - \beta_1^2 \sigma_m^2 = 0.028$$

- $\sigma_2^2 = 0.1$

$$\beta_2^2 \sigma_m^2 = 0.032$$

$$\sigma_{\epsilon_2}^2 = \sigma_2^2 - \beta_2^2 \sigma_m^2 = 0.068$$

- Although assets 1 and 2 have the same variance, they don't have the same level of systematic risk.
- The variance criteria would conclude that the two assets have the same risk when asset 1 has more systematic risk than asset 2.
- This will be reflected in the price of these two assets.

Questions

1. Can a beta of an asset be negative?

- YES
- A negative beta means that the return on an asset moves in opposite direction to the return of the market.
- This type of asset offers protection against downturn in the economy because it pays when the market goes down.
- See article below.

2. If so, what is the implication on its expected return?

Consider the CAPM

$$E(\tilde{r}_i) = r_f + \beta_i [E(\tilde{r}_m) - r_f]$$

- If asset i has a negative beta, it means that $E(\tilde{r}_i) < r_f$. Thus, this type of risky asset offer a lower return than the risk-free rate.
- This asset may not be interesting on its own but could be quite useful in a strategy of portfolio diversification in order to reduce the beta of a portfolio.

Can betas be negative? (and other well used interview questions)

Here is a favorite question among corporate finance interviewers: "Can betas be negative? And if so, what exactly do they tell us?" The reason negative betas pose a conundrum to many finance students is that they seem to go against intuition. After all, if a beta of one is average risk and a beta of zero is riskless, how can an investment have negative risk?

Here is the answer. Yes, betas can be negative. To see how and why, consider what betas measure: the risk added by an investment to a well diversified portfolio. By that definition, any investment that when added to a portfolio, makes the overall risk of the portfolio go down, has a negative beta. A more intuitive way of thinking about this is that a negative beta investment represents insurance against some macro economic risk that affects the rest of your portfolio adversely. A standard example that is offered for a negative beta investment is gold, which acts as a hedge against higher inflation (which devastates financial investments such as stocks and bonds). It is also true that puts on stocks and selling forward contracts against indices will have negative betas.

What are the consequences of a negative beta? The expected return on that investment will be less than the riskfree rate; the nominal returns on gold over the last 40 years have been 2% less, on average annually, than the riskfree rate. However, that makes complete sense if you think of it as buying insurance. You are paying for the insurance by settling for a very low or even negative return.

Are there actual investments out there that have negative betas? I know that there are stocks with negative regression betas, but those are the mostly the result of something strange happening during the period of the regression - an extended lawsuit or acquisition battle throwing off the correlation with the market - rather the true betas. In fact, in my fifteen years of updating betas by sector, I have still not found a sector with a negative beta. Furthermore, even assets that, in theory, could have negative betas (gold, for instance) seem to have positive betas when securitized (gold shares, gold ETF). There seems to be something about the securitization process that makes real assets behave more like financial assets.

Aswath Damodaran, NYU, Feb. 2009

Def: The **Alpha** of an asset is equal to the difference between its expected return and the expected return corresponding to the securities market line of the CAPM:

$$\alpha_s = E(r_s) - (r_f + \beta_s[E(r_m) - r_f])$$

In equilibrium, all assets should have an alpha equal to zero.

Example¹: Let $r_f = 5\%$, $E(r_m) = 12\%$ and $\beta_s = 1.3$

According to CAPM, we can write:

$$E(r_s)^{CAPM} = r_f + \beta_s[E(r_m) - r_f] = 0.05 + 1.3[0.12 - 0.05] = 14.1\%$$

Consider the actual return of asset s over the last four years.

	1	2	3	4
Actual return	15.9%	13.2%	9.7%	17.6%
Abnormal return	1.8%	-0.9%	-4.4%	3.5%

The abnormal return indicates the deviation of the observed return of the asset with respect to the equilibrium return that would be dictated by the CAPM. This abnormal return is caused by non-systematic risk factors which can sometimes affect positively or negatively the firm's performance. Now, let's verify if the above result is satisfied.

$$E(r_s) = \frac{15.9\% + 13.2\% + 9.7\% + 17.6\%}{4} = 14.1\%$$

Thus,

$$\alpha_s = E(r_s) - (r_f + \beta_s[E(r_m) - r_f]) = 0\%$$

Conclusion: On average, the abnormal return of an asset is equal to zero. This means that non-systematic risk factors cancel out over time and the expected return of an asset will correspond to the equilibrium return as specified by the CAPM.

¹See S. Lumby & C. Jones, Corporate Finance, 8th edition, South Western

CAPM: Application (In class)

Consider the following information on the market M and company J

Table

State	Probability	Market return	J return
A	0.2	-0.15	-0.25
B	0.2	0.05	-0.08
C	0.4	0.15	0.20
D	0.2	0.20	0.40

1. Based on the above information, complete the following Table

r_f	0.03
$E(r_m)$	0.08
$E(r_j)$	0.094
σ_m^2	0.0156
σ_m	0.1249
σ_j^2	0.0529
σ_j	0.2301
$\sigma_{j,m}$	
$\rho_{j,m}$	
β_j	

2. Determine the expected return of J in equilibrium.
3. Draw the Security Market Line (SML). Identify M and J .
4. Should you invest in company J ?

6.2 Beta of a portfolio

Consider a portfolio of P composed of N assets. The random return of this portfolio is equal to:

$$\tilde{r}_P = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + w_3 \tilde{r}_3 + \dots + w_N \tilde{r}_N$$

The covariance between this portfolio and the market portfolio M is equal to:

$$\sigma_{PM} = w_1 \sigma_{1M} + w_2 \sigma_{2M} + w_3 \sigma_{3M} + \dots + w_N \sigma_{NM}$$

According to CAPM,

$$\beta_1 = \frac{\sigma_{1M}}{\sigma_M^2} \quad \Rightarrow \quad \sigma_{1M} = \beta_1 \sigma_M^2$$

$$\beta_P = \frac{\sigma_{PM}}{\sigma_M^2} \quad \Rightarrow \quad \sigma_{PM} = \beta_P \sigma_M^2$$

Replacing in the covariance equation, we get:

$$\beta_P \sigma_M^2 = w_1 \beta_1 \sigma_M^2 + w_2 \beta_2 \sigma_M^2 + \dots + w_N \beta_N \sigma_M^2$$

which simplifies to:

$$\beta_P = w_1 \beta_1 + w_2 \beta_2 + \dots + w_N \beta_N$$

Conclusion: The beta of a portfolio is equal to the weighted average of the beta of the assets in the portfolio.

6.3 CAPM and Discounting

The CAPM provides an answer to the choice of the discount rate to use in capital budgeting decisions. There exists two equivalent methods:

1. Discount a risky cash flow at the rate corresponding to the risk of the cash flow,
2. Discount the certainty equivalent of a risky cash flow at the risk free rate.

Consider an asset x with a value of V_0 at $t = 0$ and $\tilde{V}_1$ at $t = 1$.

The return (ex post) and expected return (ex ante) of this asset over one period are:

$$\tilde{r}_x = \frac{\tilde{V}_1 - V_0}{V_0} = \frac{\tilde{V}_1}{V_0} - 1$$

$$E(\tilde{r}_x) = \frac{E(\tilde{V}_1) - V_0}{V_0} = \frac{E(\tilde{V}_1)}{V_0} - 1$$

Méthod 1.

According to the CAPM:

$$E(\tilde{r}_x) = r_f + \beta_x [E(\tilde{r}_m) - r_f]$$

Using the definition of the expected return of asset x over one period, we get:

$$\frac{E(\tilde{V}_1)}{V_0} - 1 = r_f + \beta_x [E(\tilde{r}_m) - r_f]$$

$$\frac{E(\tilde{V}_1)}{V_0} = 1 + r_f + \beta_x [E(\tilde{r}_m) - r_f]$$

$$V_0 = \frac{E(\tilde{V}_1)}{1 + r_f + \beta_x [E(\tilde{r}_m) - r_f]}$$

Implication: A risky cash flow must be discounted at a rate that includes a risk premium. The CAPM gives that risk premium. If the cash flow is non risky, its beta is equal to zero, $\beta_x = 0$, and the cash flow must be discounted at the risk free rate, r_f .

Method 2

We know that;

$$\beta_x = \frac{\text{Cov}(\tilde{r}_x, \tilde{r}_m)}{\sigma_m^2}$$

We can replace $\tilde{r}_x$ by its definition;

$$\beta_x = \frac{\text{Cov}(\frac{\tilde{V}_1}{V_0} - 1, \tilde{r}_m)}{\sigma_m^2} = \frac{\text{Cov}(\frac{\tilde{V}_1}{V_0}, \tilde{r}_m)}{\sigma_m^2}$$

We also know that:

$$V_0 = \frac{E(\tilde{V}_1)}{1 + r_f + \beta_x [E(\tilde{r}_m) - r_f]}$$

We can replace by the value of β_x :

$$V_0 = \frac{E(\tilde{V}_1)}{1 + r_f + \frac{\text{Cov}(\frac{\tilde{V}_1}{V_0}, \tilde{r}_m)}{\sigma_m^2} [E(\tilde{r}_m) - r_f]}$$

or alternatively,

$$V_0(1 + r_f) + \frac{\text{Cov}(\tilde{V}_1, \tilde{r}_m)}{\sigma_m^2} [E(\tilde{r}_m) - r_f] = E(\tilde{V}_1)$$

Hence,

$$V_0 = \frac{E(\tilde{V}_1) - \frac{\text{Cov}(\tilde{V}_1, \tilde{r}_m)}{\sigma_m^2} [E(\tilde{r}_m) - r_f]}{(1 + r_f)}$$

The numerator of this equation is called the **certainty equivalent** of the risky cash flow, $\tilde{x}$.

The expression $\frac{\text{Cov}(\tilde{V}_1, \tilde{r}_m)}{\sigma_m^2}$ can be interpreted as the market price of risk.

6.4 CAPM, Information and Asset Pricing

CAPM is a model which provides the equilibrium return of any asset.

In equilibrium, the price of an asset will reflect its equilibrium return.

Consider the return of a stock over one year. We can write:

$$E(\tilde{r}_{i,t}) = \frac{E(\tilde{D}_{i,t+1}) + E(\tilde{P}_{i,t+1}) - P_{i,t}}{P_{i,t}}$$

Suppose the following information on asset i :

- $P_{i,t} = 500$
- $E(\tilde{D}_{i,t+1}) = 20$
- $E(\tilde{P}_{i,t+1}) = 530$

Based on this information, we can calculate the expected return over the year.

$$E(\tilde{r}_{i,t}) = \frac{20 + 530 - 500}{500} = 0.10$$

Suppose that you receive new information on the firm which just signed a new big contract in Asia and expects to increase its profits in the coming years. Based on this, you update the previous expectations such that $E(\tilde{D}_{i,t+1}) = 30$ and $E(\tilde{P}_{i,t+1}) = 540$

Question: What happens then? Is the expected return of the asset affected? Is the price of the asset affected?

Answer: CAPM can help us understand the impact of this new information.

- According to CAPM, the expected return of an asset depends only on its *beta*.
- If the *beta* does not change, the expected return does not change.

- An increase in profit will affect the cash flows of the firm but typically not its systematic risk.
- If so, the expected return does not change. Thus, the new information will impact on the price of the stock in the following way:

$$0.10 = \frac{30 + 540 - P'_{i,t}}{P'_{i,t}} \quad \Rightarrow \quad P'_{i,t} = 518.18$$

- The price of the stock will increase to 518.18.

Question: Why is the price increase equal to 18.18?

Answer: ???

NOBEL PRIZE

William Sharpe on the CAPM

William Sharpe received the Nobel Prize in 1990 for his development of the Capital Asset Pricing Model. Here are his comments on the CAPM from a 1998 interview with Jonathan Burton:*

Portfolio theory focused on the actions of a single investor with an optimal portfolio. I said, What if everyone was optimizing? They've all got their copies of Markowitz and they're doing what he says. Then some people decide they want to hold more IBM, but there aren't enough shares to satisfy demand. So they put price pressure on IBM and up it goes, at which point they have to change their estimates of risk and return, because now they're paying more for the stock. That process of upward and downward pressure on prices continues until prices reach an equilibrium and everyone collectively wants to hold what's available. At that point, what can you say about the relationship between risk and return? The answer is that expected return is proportionate to beta relative to the market portfolio.

The CAPM was and is a theory of equilibrium. Why should anyone expect to earn more by investing in one security as opposed to another? You need to be compensated for doing badly when times are bad. The security that is going to do badly just when you need money when times are bad is a security you have to hate, and there had better be some

redeeming virtue or else who will hold it? That redeeming virtue has to be that in normal times you expect to do better. The key insight of the Capital Asset Pricing Model is that higher expected returns go with the greater risk of doing badly in bad times. Beta is a measure of that. Securities or asset classes with high betas tend to do worse in bad times than those with low betas.

The CAPM was a very simple, very strong set of assumptions that got a nice, clean, pretty result. And then almost immediately, we all said: Let's bring more complexity into it to try to get closer to the real world. People went on—myself and others—to what I call “extended” Capital Asset Pricing Models, in which expected return is a function of beta, taxes, liquidity, dividend yield, and other things people might care about.

Did the CAPM evolve? Of course. But the fundamental idea remains that there's no reason to expect reward just for bearing risk. Otherwise, you'd make a lot of money in Las Vegas. If there's reward for risk, it's got to be special. There's got to be some economics behind it or else the world is a very crazy place. I don't think differently about those basic ideas at all.

*Jonathan Burton, “Revisiting the Capital Asset Pricing Model,” *Dow Jones Asset Manager* (May/June 1998): 20–28.

Bank of De Mango, 2017

Calculation and publication of index value CAC40 index

The current index value is published on a continuous basis and published every 30 seconds, using real time last traded prices from Euronext Cash markets. Index values can be qualified as indicative, or the publication of index values may be suspended if there are circumstances preventing correct calculation.

The official index opening is calculated by using each opening price or the previous day reference prices in case constituents have a non-traded, halted or suspended status. In case of IPOs, buyouts and swap offers estimated prices are being used. The official index opening level reflects the opening prices of the index constituents at the respective and various times that each constituent opening may occur. The official index opening level will be confirmed on the close of the trading day.

The official opening index level is only included in the calculation of the high and low index values provided in the Euronext closing data-feed and only when the official opening index level creates the high or the low of the day.

The opening index level is disseminated at the same time as the first index level, and it might not be equal to the first index level. The official opening index level will be confirmed on the close of the trading day. The first index level will be disseminated as soon as the following conditions are met:

1. The total weight of stocks traded, halted or suspended are equal to 100% of the total free float market capitalization of the CAC 40 index.
2. The weight of stocks traded is superior or equal to 65% of the total free float market capitalization of the CAC 40 index.

When not all opening conditions have been met the CAC40 index can be in forerunner-state. In case the CAC40 is in forerunner-state at the time of closing, the closing level will be as a consequence calculated on the basis of the most recent traded prices, or the most recent reference price. During the session, the forerunner can be substituted for the index on an exceptional basis in the event that 35% of the market capitalization of the index may not be traded. The calculation and dissemination of the index levels will resume five minutes after the weight of the traded stocks is superior or equal to 65% of the free float market capitalization of the index.

The last published level of the CAC40 index will be the official closing level of the CAC40 index for the relevant trading day. The index value is calculated as the result of the basis level multiplied by the sum of the prices multiplied by the corresponding weights, and divided by the adjustment coefficient multiplied by the basis capitalisation.

$$I_t = 1000 \times \frac{\sum_{i=1}^N Q_{i,t} F_{i,t} f_{i,t} C_{i,t}}{K_t \sum_{i=1}^N Q_{i,0} C_{i,0}}$$

Where:

t	day of calculation
N	number of constituent equities in index
$Q_{i,t}$	number of shares of equity i on day t
$F_{i,t}$	free float of equity i
$f_{i,t}$	capping factor of equity i
$C_{i,t}$	price of equity i on day t
$Q_{i,0}$	number of shares of equity i on index base date
$C_{i,0}$	price of equity i on index base day
K_t	adjustment coefficient for base capitalization on day t

Example 10.8

Estimating Beta

Problem

Suppose the market portfolio excess return tends to increase by 47% when the economy is strong and decline by 25% when the economy is weak. What is the beta of a type S firm whose excess return is 40% on average when the economy is strong and -20% when the economy is weak? What is the beta of a type I firm that bears only idiosyncratic, firm-specific risk?

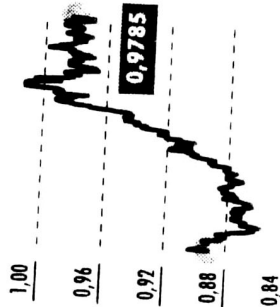
Example 10.8 (cont'd)

Solution

The systematic risk of the strength of the economy produces a $47\% - (-25\%) = 72\%$ change in the return of the market portfolio. The type S firm's return changes by $40\% - (-20\%) = 60\%$ on average. Thus the firm's beta is $\beta_S = 60\% / 72\% = 0.833$. That is, each 1% change in the return of the market portfolio leads to a 0.833% change in the type S firm's return on average.

The return of a type I firm that has only firm-specific risk, however, is not affected by the strength of the economy. Its return is affected only by factors that are specific to the firm. Whether the economy is strong or weak, it will have the same expected return, and thus $\beta_I = 0\% / 72\% = 0$.

L'EURO EN DOLLAR



La Tribune

Source: Bloomberg

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PHOTO: AFP/REUTERS

► PLACEMENTS Jouer le « bêta » pour la volatilité

- Morgan Stanley recommande de se placer sur les secteurs à bêta élevé.
- A savoir les nouvelles technologies et la finance.

luent donc à l'inverse du CAC. Les secteurs recommandés par Morgan Stanley ? Les technologies de l'information et la finance... Deux domaines qui, aujourd'hui, ne sont guère prisés. « Nous escomptons une remontée des marchés d'actions d'ici à la fin de l'année, sous l'effet d'une reprise de l'économie mondiale et d'une amélioration des résultats des entreprises », justifie la banque.

Se pencher sur les bêta. » C'est par cette formule énigmatique que la banque américaine Morgan Stanley, dans une étude diffusée en septembre, appelle les investisseurs à se positionner sur les valeurs qui amplifient les mouvements de la Bourse. Une invitation fondée sur l'hypothèse d'un prochain rebond des marchés.

Vivendi Universal illustre bien l'intérêt des bêta. Si l'indice CAC 40 grimpe de 1 %, le titre du groupe de médias - doté d'un bêta de 1,74, le plus élevé de l'indice CAC 40 selon le consensus établi par Jacques Chahine Finance Group (JCF Group) - gagne, en théorie, 1,74 %. A l'opposé, certaines mines d'or affichent un bêta négatif. Elles évo-

Mode d'emploi

Le bêta d'une action mesure sa volatilité. Il permet donc de tirer parti des fluctuations de la Bourse.

Les valeurs dotées d'un bêta supérieur à 1 amplifient, en théorie, les variations du marché.

Les titres dont le bêta est compris entre 0 et 1 progressent à l'image de la Bourse.

Les actions assorties d'un bêta négatif évoluent à l'inverse du marché.

dérivage, le CAC 40 peinera à regagner 20 à 30 % », reconnaît Jean-Noël Vieille, stratège chez Aurel Leven.

Reste que le manque de visibilité sur l'activité décourage d'investir. Exception faite de Thomson Multimedia (bêta de 1,22), selon Jean-Noël Vieille. A l'inverse, l'équipementier de télécommunications Alcatel a beau bénéficier d'un bêta de 1,67, le stratège reste à l'écart de la valeur. Il faut dire que, d'après les analystes financiers de Credit Suisse First Boston, le marché mondial des équipements de télécommunications devrait chuter de 36 % en 2002, pour se limiter à 16 milliards d'euros. Si bien qu'Alcatel risque d'accuser 900 millions d'euros de pertes cette année.

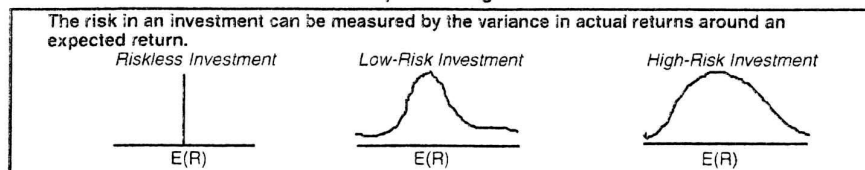
« Je continue de préférer les secteurs défensifs », conclut Jean-Noël Vieille. Quitte à manquer le train du regain de la Bourse. Le groupe de distribution Carrefour et le cimentier Lafarge - archétypes de valeurs défensives - présentent des bêta respectivement limités à 0,98 et 0,63.

Christine Lejoux

lectionner des valeurs, « le seul bêta ne suffit pas. Il sert essentiellement à corroborer un choix ». nuance Loïc de Galzain, analyste technique chez SG Securities. Certes, les valeurs de télécommunications et de services informatiques devraient rebondir davantage que le marché, leurs bêta dépassant généralement 1, selon des statistiques observées sur longue période. « Sans les technologiques, responsables de son

www.latribune.fr

Step 1: Defining Risk



Step 2: Differentiating between Rewarded and Unrewarded Risk

Risk that is specific to investment (firm-specific risk) can be diversified away in a diversified portfolio. 1. Each investment is a small proportion of portfolio. 2. Risk averages out across investments in portfolio. The marginal investor is assumed to hold a diversified portfolio. Thus, only market risk will be rewarded and priced.	Risk that affects all investments (market risk) cannot be diversified away since most assets are affected by it.
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Step 3: Measuring Market Risk

The CAPM	The APM	Multifactor Models	Proxy Models
If there is 1. no private information 2. no transactions cost the optimal diversified portfolio includes every traded asset. Everyone will hold this market portfolio. Market Risk = Risk added by any investment to the market portfolio	If there are no arbitrage opportunities, then the market risk of any asset must be captured by betas relative to factors that affect all investments. Market Risk = Risk exposures of any asset to market factors	Since market risk affects most or all investments, it must come from macroeconomic factors. Market Risk = Risk exposures of any asset to macro-economic factors	In an efficient market, differences in returns across long periods must be due to market risk differences. Looking for variables correlated with returns should then give us proxies for this risk. Market Risk = Captured by the proxy variable(s)
Betas of assets relative to market portfolio (from a regression)	Betas of assets relative to unspecified market factors (from a factor analysis)	Betas of assets relative to specified macro-economic factors (from a regression)	Equation relating returns to proxy variables (from a regression)

FIGURE 4.5 Risk and Return Models in Finance

Source: Damodaran 2002

Rentabilité ou rendement des actions et des obligations ?

Dans deux articles parus dans la presse économique, le fait que les actions rapportent plus que les emprunts d'Etat (sovereign bonds) ou que les dettes des entreprises (corporate bonds) a été présenté comme une « anomalie » historique. Les auteurs en concluent que les actions sont sous-évaluées, voire « massacrées », et qu'il est donc avisé, et même urgent, de les acheter. Leur analyse semble suffisamment partagée dans l'opinion pour qu'il ait paru utile d'y apporter clarification et correction. Bien que nous utilisions à titre illustratif des éléments chiffrés particuliers relatifs à l'actualité, notre propos se veut général.

La recommandation des auteurs est basée sur le raisonnement suivant : le rendement moyen projeté des actions du CAC40, mesuré par le taux de dividende 2010 (dividende / valeur boursière), est supérieur de 1,3% à celui des emprunts d'Etat (OAT) à 10 ans, soit 4,39% contre 3,09%. Comme, sur les 15 dernières années, le différentiel moyen de rendement a été négatif de -1,6%, le rendement relatif des actions du CAC40 est supérieur de 2,9% à ce qu'il devrait être « normalement ». Il y a donc sous-évaluation des actions.

Le même raisonnement est appliqué aux actions européennes du Stoxx600 : le taux de dividende moyen projeté pour 2010 est de 3,9% contre 2,5-3% en moyenne historiquement, contre 2,6% de rendement sur les Bunds allemands à 10 ans, 3,1% sur les OAT à 10 ans ou encore 3,35% sur les dettes des entreprises européennes cotées AAA. Les actions semblent donc, encore une fois, sous-évaluées.

Or, tout investisseur sait bien que les actions sont plus risquées que les obligations « corporate », elles mêmes plus risquées (en général, les emprunts souverains des PIIGS nonobstant) que les dettes des Etats. De ce fait, elles font l'objet d'une prime de risque plus élevée, comme l'enseigne tout cours de base de finance. Les précédents auteurs commettent en fait deux erreurs majeures de raisonnement, à savoir confondre rendement et rentabilité d'une part, et rentabilité espérée (ou « ex ante ») et rentabilité constatée (« ex post ») d'autre part. Notons que par abus de langage, nous reprenons comme les auteurs la terminologie imprécise « rendement » et « rentabilité » en lieu et place de « taux de rendement » et « taux de rentabilité ».

En ce qui concerne le premier point, ce qui compte pour l'investisseur est la rentabilité du titre (action ou obligation). Or, cette *rentabilité* comprend non seulement le dividende ou le coupon, ce qui détermine le *rendement* du titre, mais également la plus ou moins value en capital (qui d'ailleurs représente en général la part déterminante pour les actions). Ceci explique en particulier pourquoi deux firmes de risque identique peuvent offrir la même rentabilité espérée alors qu'elles affichent des politiques de distribution de dividendes très différentes. Etablir des comparaisons sur la seule base des rendements n'a pas de sens.

Pour ce qui est du deuxième point, il est peut-être encore plus grave de confondre rentabilités « ex ante » et « ex post ». La finance nous enseigne que pour inciter les investisseurs à prendre des risques, il faut les rémunérer par des primes de risque « ex ante ». Par exemple, un titre offrira une rentabilité espérée d'autant plus élevée qu'il est

risqué (c'est le cours actuel du titre qui s'ajuste, à la hausse ou à la baisse, pour réaliser cela). Comme les actions sont plus risquées que les obligations, elles doivent nécessairement offrir une rentabilité espérée plus élevée. Dans le cas contraire, personne ne les achèterait. Le prix du risque dépend évidemment des pays et des circonstances et varie avec le temps en fonction de la relative frilosité et de la richesse des investisseurs. Par exemple, la prime de risque ex ante sur le marché américain des actions se situe en moyenne séculaire autour de 5%, c'est-à-dire que les actions US offrent, en espérance, 5% de plus que le taux sans risque que l'on mesure généralement par le taux des Bons du Trésor. Mais naturellement, ex post, la prime de risque peut être négative. Pis encore, la rentabilité constatée (avant déduction du taux sans risque) peut également l'être, quand le marché a été suffisamment baissier pour effacer les gains en dividendes. C'est d'ailleurs en cela que le risque existe.

En conclusion, ce n'est que si l'on compare les rentabilités espérées des actions et celles des obligations en tenant compte également de leur niveau de risque de marché (mesuré par exemple par le bêta) que l'on peut se faire une idée de la cherté relative des unes et des autres. Il serait intéressant de savoir si les auteurs cités en début d'article ont mis en application leur conseil et se sont rués pour acheter massivement des actions soi-disant « massacrées ». Rien n'est moins sûr !

Jocelyn Martel

Patrice Poncet

7 Market Efficiency

The question of market efficiency is probably one of the most important issue in finance. Although this notion has been used to denote different phenomena, and many times wrongfully so, it has a very specific meaning in finance.

Question : What is meant by *market efficiency* ?

Answer: We know that the price of securities reflect the expectations of agents on the market with respect to their futur returns. Thus, we say that markets are **efficient** when security prices reflect all available information.

This rests on two important assumptions:

1. investors are price-takers (competitive markets)
2. instantaneous circulation of information and at no costs

Note: In reality, information is costly so a more realistic definition of market efficiency is that prices reflect all information until the marginal cost of information (including trading cost), commonly called transaction costs, is equal to the margainal benefit of acquiring this information.

7.1 Some background facts

The idea of market efficiency goes back to the 1950's where academics (economists and statisticians) started examining the behavior of stock prices. Consider the following graph (Brealy & Myers). A first one depicts the evolution of wealth of a gambler who plays "head or tail". Starting from an initial wealth of \$100, he wins 3% if he gets "head" and loses 2.5% if he gets "tail". A second graph depicts is the evolution of the S&P 500 over a 5 year period.

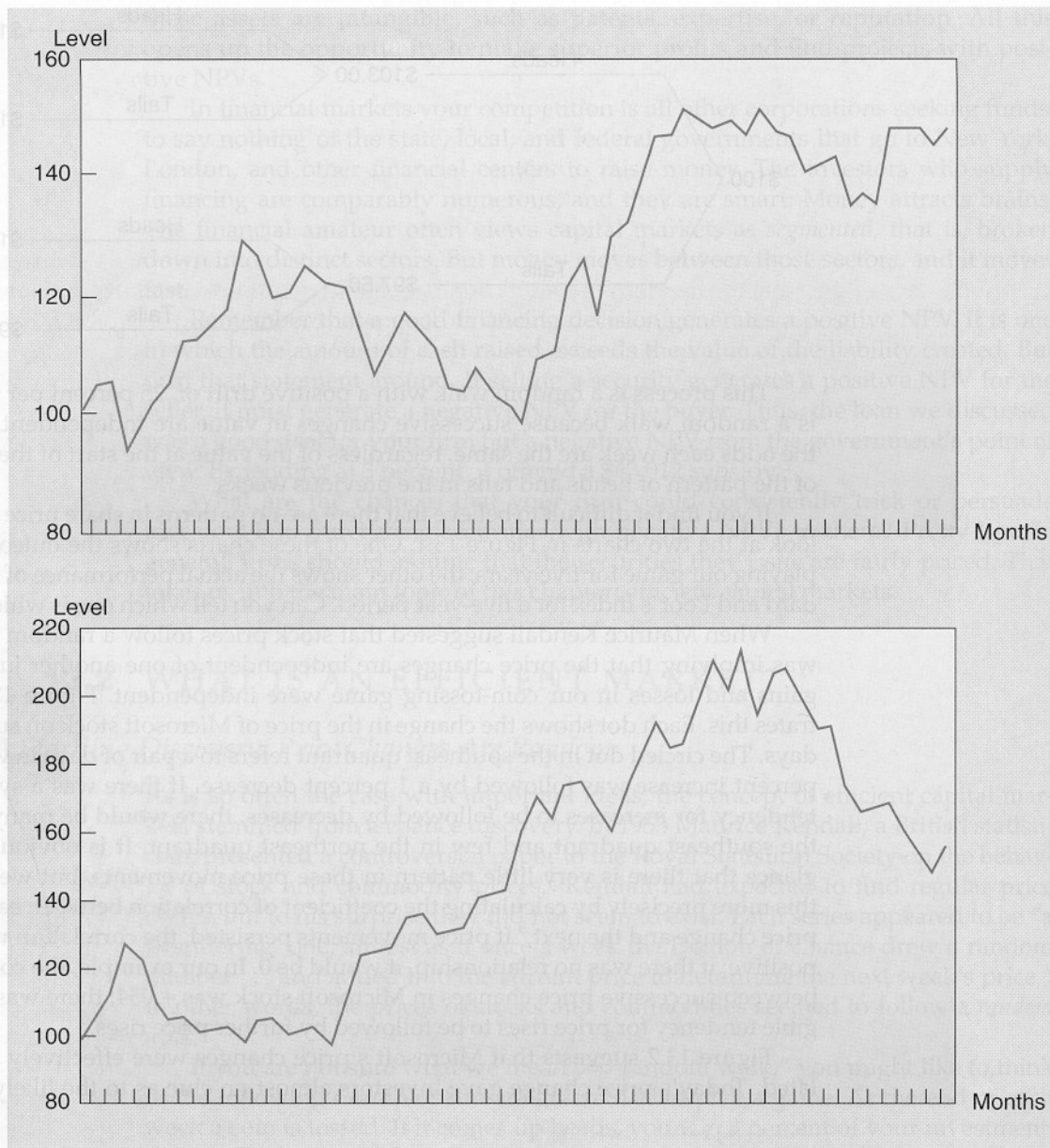


Figure 13.1

One of these charts shows the Standard and Poor's Index for a five-year period. The other shows the results of playing our coin-tossing game for five years. Can you tell which is which?

Question: Which graph depicts the random game and which depicts the stock market?

Answer: Top graph: stock market

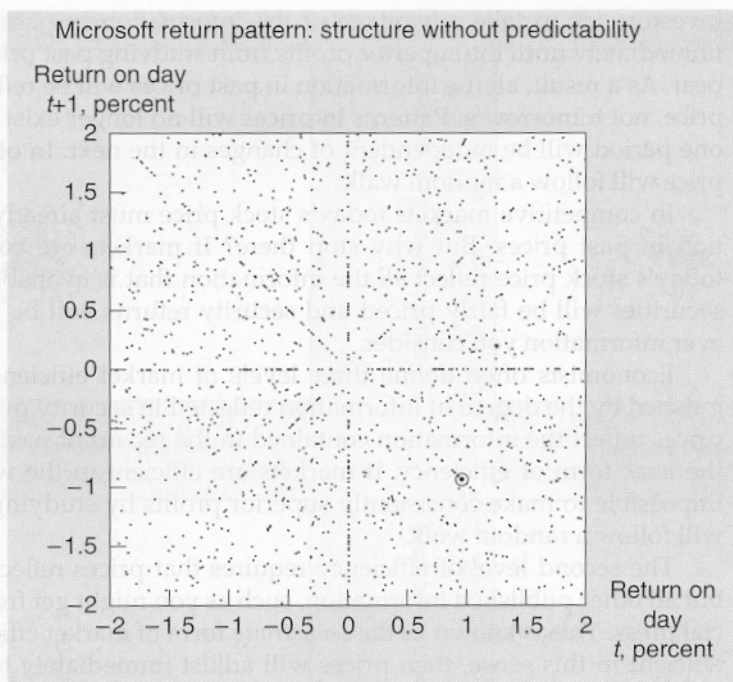
Conclusion : At first look, the stock market seems to behave like a random game and the prices seems to follow a random walk, that is the variation in prices are independent over time.

One argument which is made against this conclusion is that an increase in the price of a stock is typically followed by a decrease and vice-versa. Below is the evolution of Microsoft stock price from t to $t+1$.

The following graph illustrates this notion of time independence (Brealy & Myers).

Figure 13.2

Each dot shows a pair of returns for Microsoft stock on two successive days between August 1993 and August 1998. The circled dot records a daily return of +1 percent and then -1 percent on the next day. The scatter diagram shows no significant relationship between returns on successive days.



If price increase were followed by price decrease and vice-versa, most points would be in quadrants to the north-west and south-east. This is not the case. The graph suggests that any price sequence is as possible as any other. Therefore, the price of Microsoft at t cannot be used to predict the price at $t+1$.

Question: Is this surprising?

Answer: NO! If investors could use past prices to predict future prices, anyone could become unimaginably rich.

7.2 Definition of efficiency

The notion of market efficiency implies that prices reflect all available information. The question is what type of information. This leads up to three definitions of efficient markets (Fama, 1988).

1. Weak efficiency

- Current prices reflect all available information contained in historical prices.
- If so, any change in prices can only result from unexpected events.
- No investor can earn an excess return (*beat the market*) based on the observation of historical prices.

2. Semi-strong efficiency

- Current prices reflect all available information contained in historical prices as well as all publicly available information concerning the security (i.e. earnings announcement,

new equity issue, strategic decisions, macroeconomic factors, expectations on fiscal and monetary policy, ...).

- No investor can earn an excess return (*beat the market*) based on historical prices and public information.
- The only possibility to earn an excess return is to use privately held information.

3. Strong efficiency

- Current prices reflect all available information contained in historical prices, all publicly available information as well as privately held information concerning the security.
- Impossible for any investor to earn an excess return (*beat the market*).

Before looking at the implications of market efficiency, let's clarify one important point. There are two questions related to the integration of information in security prices:

1. How fast information is incorporated in security prices.
2. Do security prices correctly reflect the fundamental information.

The first question refers to *market efficiency*. The second question one refers to *market rationality*.

Illustration of market efficiency:

- Suppose that Nestle announces that, unlike what had been announced at the beginning of the year, earnings will be larger than expected without any additional growth.
- This increase in earnings is due to internal restructuring and considered as permanent.
- Investors believe that this announcement is credible.
- If market are efficient, the stock price of Nestle should go up upon the earnings announcement in order to reflect this increase in the firm's value.

Question: Is it possible for an investor to earn an excess return on this security?

Answer: Consider the following scenario.

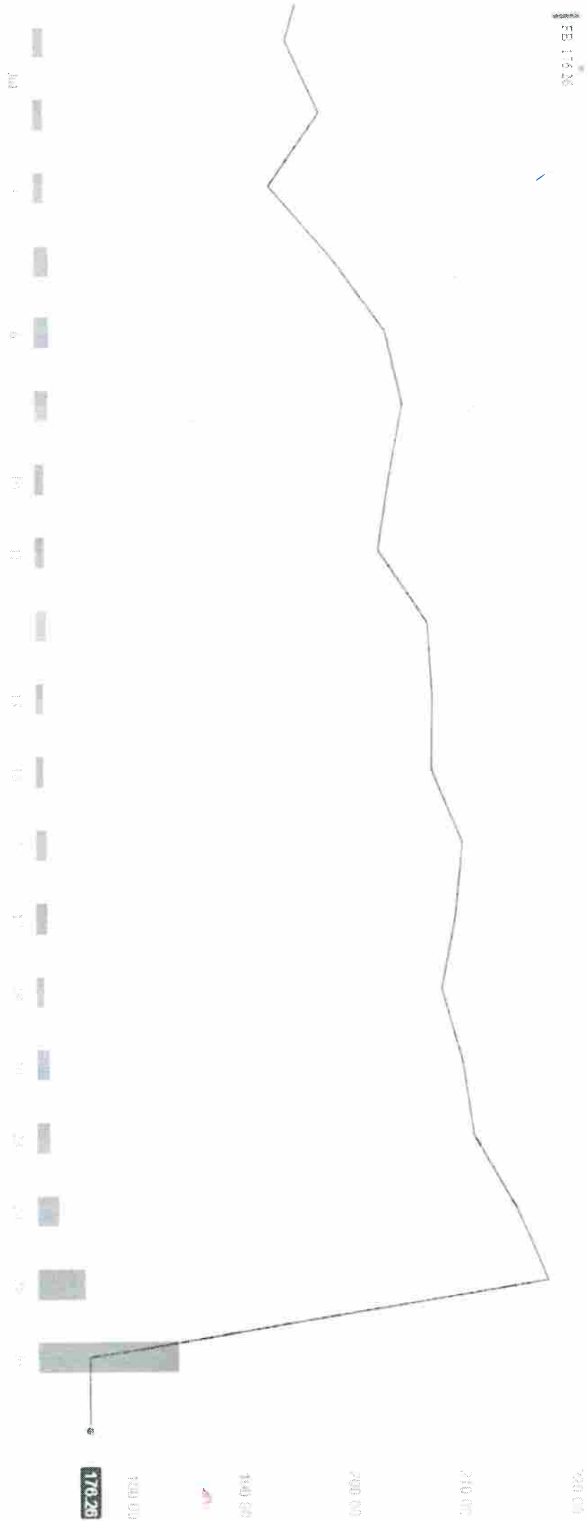
- Expecting a price increase, an investor buys shares immediately after the announcement.
- According to the semi-strong form of market efficiency, all investors, buyers and sellers of the security, will reassess the value of that stock based on the new information.
- Given that earnings are expected to increase, this leads to an immediate increase in price.
- Yet, the new price may not immediately be the equilibrium price as it may take some time for investors to fully evaluate the impact of this information on the firm's value and stock price.
- Thus, by buying immediately after the announcement, an investor may be paying too little or too much compared to the equilibrium price.

- If markets are efficient in the semi-strong form, on average, an investor would be paying the correct price for the security.
- Thus, on average, an investor cannot earn an excess profit based on this announcement.

Conclusion: New information about securities are dissipated very quickly on financial markets and should be rapidly reflected in prices.

Some examples.

Facebook July 26, 2018



One Day Loss : \$ 119 Billion

Ericsson Speeds Cost Cuts as Turnaround Proves Challenging

By **Niclas Rolander**

18 juillet 2017 à 07:46 UTC+2

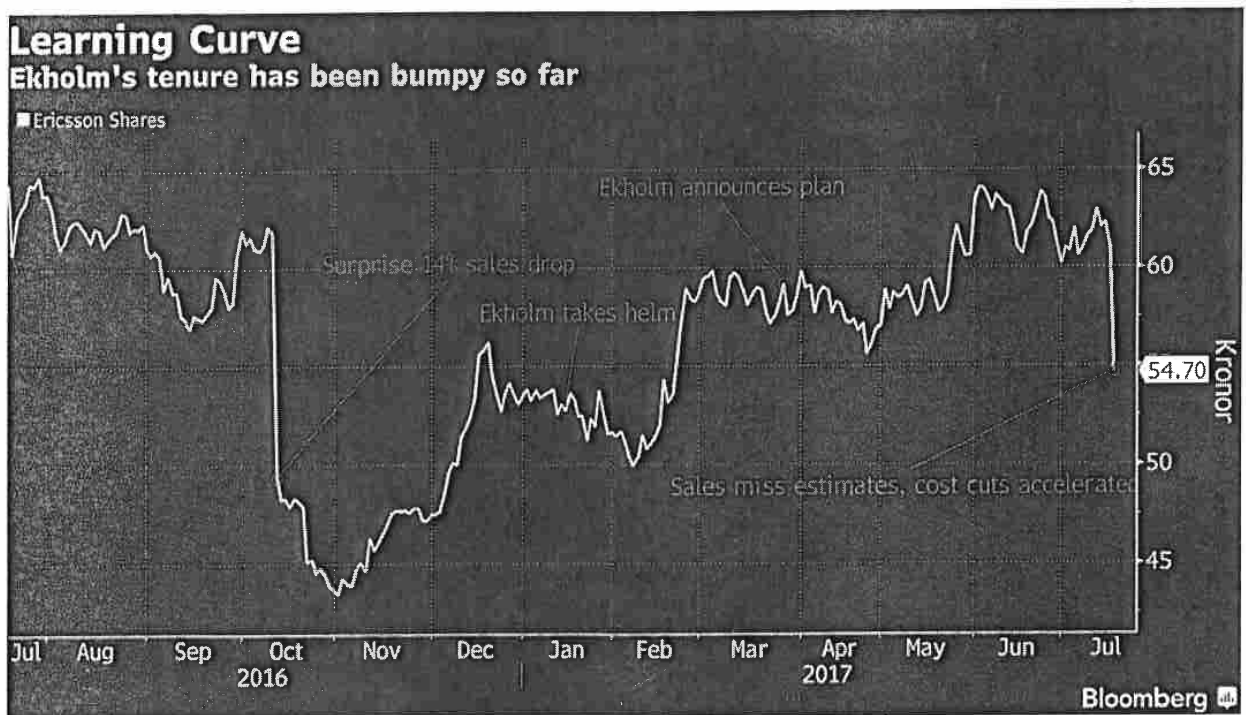
Updated on 18 juillet 2017 à 14:21 UTC+2

- Shares fall most in nine months as earnings miss estimates
- CFO Mellander asks for patience: 'there is no quick fix'

Ericsson AB <<https://www.bloomberg.com/quote/ERICB:SS>> cautioned that turning around the beleaguered phone-equipment maker will require even steeper cost cuts, testing the patience of investors who sent the stock tumbling the most this year.

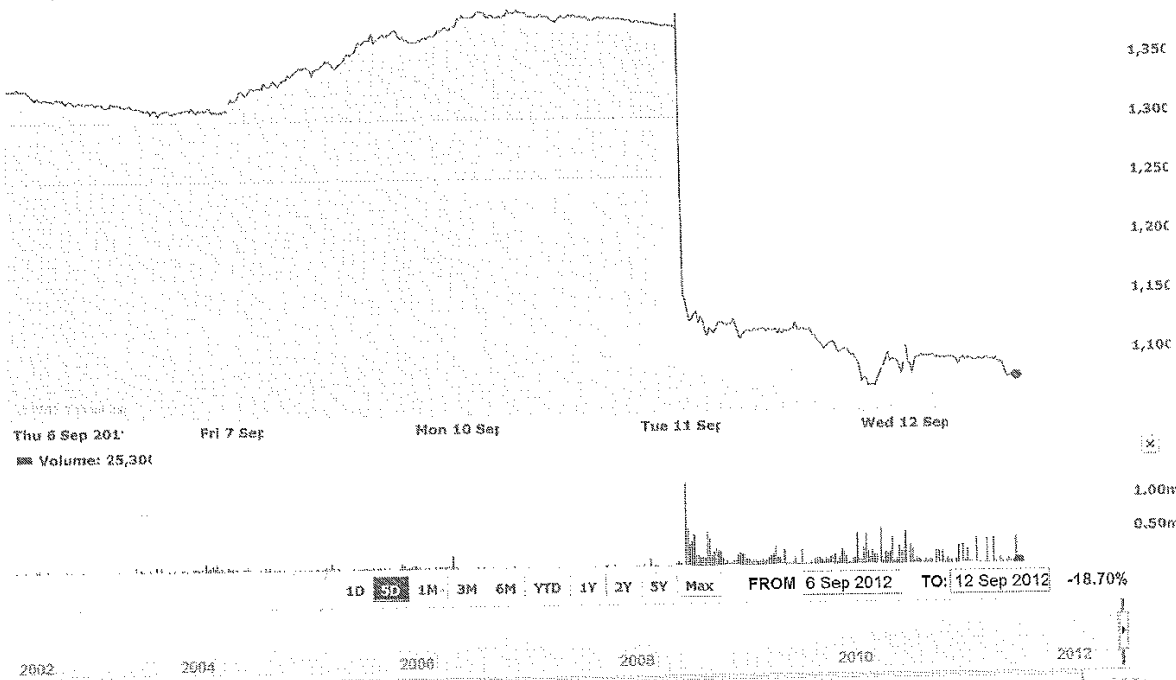
Ericsson fell as much as 13 percent in Stockholm after the company posted a second-quarter loss and warned that a faltering market amid technology shifts could cause as much as 5 billion kronor (\$600 million) of operating income to evaporate over the next 12 months.

"We're three months into the strategy," Chief Financial Officer Carl Mellander said in an interview. "You should have patience because this is about value creation and the long game. There is no quick fix, this is about strategic work and a repositioning of the company."



Chief Executive Officer Borje Ekholm, who took the helm in January, is under pressure from activist investor Christer Gardell to deliver a speedy turnaround. His plan hinges on reducing costs and scaling back expansion plans that haven't panned out, while refocusing on Ericsson's core business of selling networking equipment ahead of the expected roll-out of 5G networks. Gardell's fund Cevian has acquired about a 6 percent stake in the company since March.

12 Sep 2012 12:17 - 12:24 35" BRBY.L 1078.83



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WEDNESDAY, SEPTEMBER 12, 2012

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Burberry Sends Warning Signal

By PAUL SONNE
AND PETER EVANS

LONDON— **Burberry Group** PLC issued a surprise profit warning on Tuesday and reported its worst same-store sales figures since the doldrums of the financial crisis, raising concerns about a slowdown in the luxury sector, where companies are feel-

ing the pangs of a decelerating China and macroeconomic uncertainty world-wide.

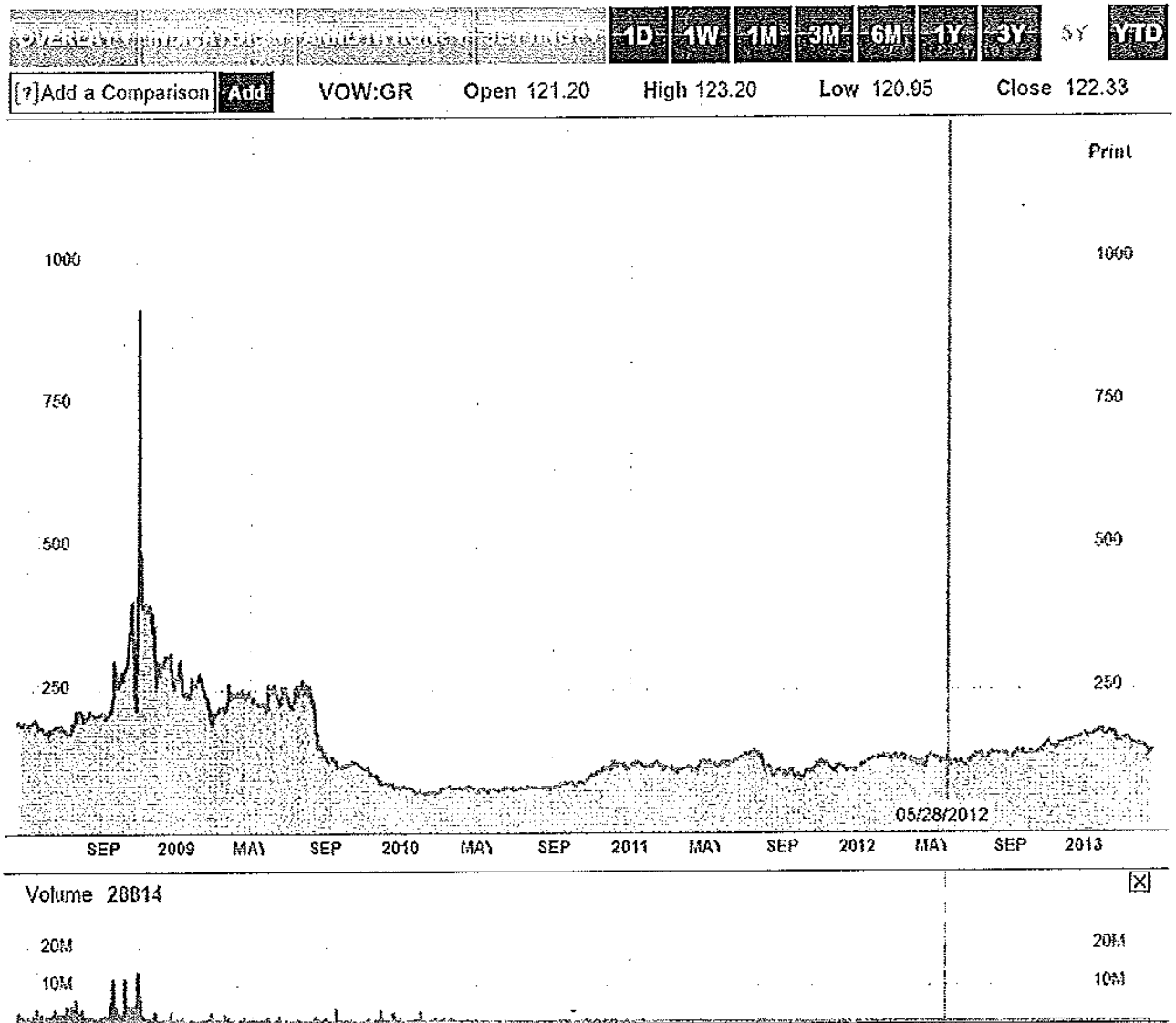
The British fashion house, known for its classic trench coats and Haymarket check, is scheduled to report full quarterly sales figures in October. But it divulged Tuesday a slump in sales growth for the quarter's 10 weeks to date, ahead of investor meetings

this month.

It said sales at Burberry stores that have been open at least a year were flat compared with a year earlier—the worst performance since 2008—though sales increased 6% when including new store space. Burberry also warned that pretax profit for the fiscal year ended March 31 would come in at “the lower

end of market expectations,” or between £407 million and £455 million (\$651 million and \$727 million).

The announcement sent Burberry's stock down 21% to £10.88 in London trading and ricocheted across the luxury sector, hitting shares of **PPR SA**, **LVMH Moët Hennessy Louis Vuitton SA** and Cie. **Financiere Richemont SA**.



Volkswagen

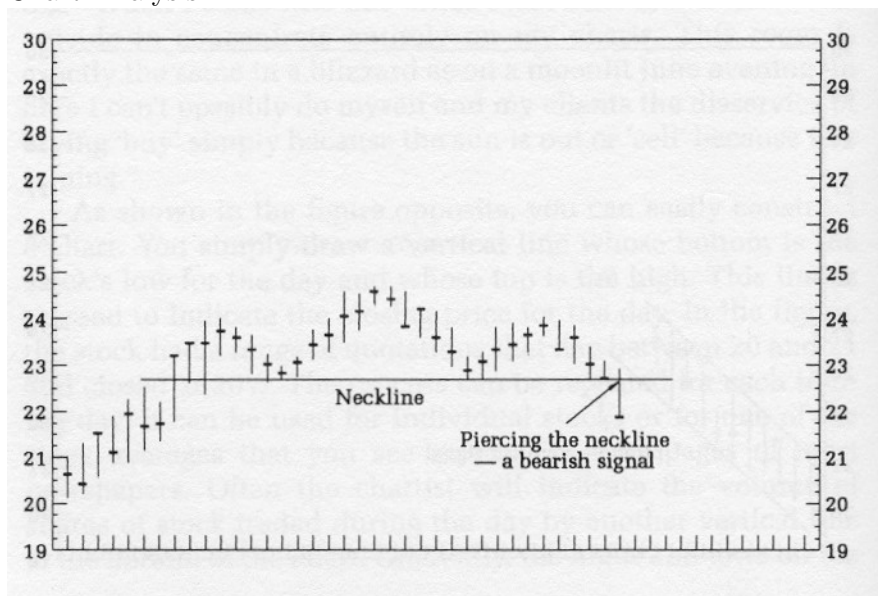
7.3 Implications of market efficiency

Market efficiency does not imply that the value of a security is equal to its right value at all times. It says that there is no way to use the information on a security at time t to earn an abnormal return, that is a profit over and above what would be dictated by the risk of the security (CAPM, Arbitrage Theory, Multi-factors models....).

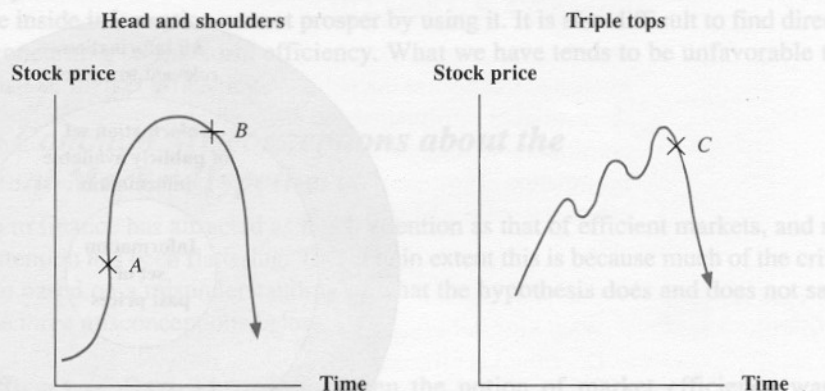
The efficient market hypothesis has many important implications.

1. If markets are weakly efficient, *Technical analysis* (commonly called *Chart analysis*) (see graphs below), which rests on finding recurrent trend and patterns in security prices, is worthless.
2. If markets are semi-strong efficient, *Fundamental analysis*, which calculate the value of a security from its fundamentals (earnings, risk, industry, macro-environment, ...), is doomed to fail. This type of analysis is used extensively by equity analyst in the recommendations to buy or sell a stock. In addition, this would mean that a passive investment strategy such as *buy and hold* dominates an active investment strategy which rests on security analysis and stock picking (see below for a discussion of the role of active portfolio managers).
3. It is generally acknowledged that markets are not strongly efficient. Indeed, any investor with privileged information should be able to earn an abnormal profit. Yet, this is illegal and considered as “insider trading”.

Chart Analysis:



■ **FIGURE 13.3 Two Widely Believed Technical Patterns**



Technical analysts frequently claim that the price of a stock is likely to follow a head-and-shoulders pattern or a triple-tops pattern. According to a technical analyst, if a head-and-shoulders pattern can be identified early enough, an investor might like to buy at point A and sell at point B. A triple-tops pattern occurs when three highs are followed by a precipitous drop. If a triple-tops pattern can be identified early enough, an investor might like to sell at point C.

7.4 Tests of market efficiency

Market efficiency is an hypothesis. Whether or not it prevails is a matter of empirical investigation.

7.4.1 Return predictability

Question: can we predict the return of a security from past data?

1. Time pattern in returns (correlation and run tests)

- Consider a regression model

$$r_t = a + b r_{t-1-T} + \epsilon_t$$

- Using past information ($T=0, 1, 2, \dots$), we estimate the correlation between r_t and r_{t-1-T} .
- Results from studies: Although there seems to be some positive relationship between returns over a short period, the magnitude is typically very small and very little of today's return is explained by yesterday's return.
- Other types of tests such as run tests (look at the sequence of signs in the variation of a stock price) arrive at the same conclusion as the correlation tests.
- Other tests have looked at long term correlation. Although some negative correlations (long-run reversal patterns) have been found, the results are not conclusive and can have many interpretations.
- Strong evidence in support of weak efficiency.
- One caveat is the existence of a *momentum effect*. According to some studies, good and bad performances of some stocks continue over time. Therefore, it seems possible to build portfolios of best-performing stocks in the recent past and generate excess return with some reliability.

7.4.2 Events studies

Idea: Examine the impact of a particular event of the security price.

1. Step 1: Use a benchmark model to estimate what the stock return would have been without the event.
2. Step 2: Calculate the stock return with the event.
3. Step 3: The **abnormal return** is equal to the difference between the actual return and the benchmark.

Suppose that we use a one factor “market model” to estimate the return of a stock. We can write:

$$\tilde{r}_t = a + b \tilde{r}_{m,t} + \tilde{\epsilon}_t$$

Thus, the abnormal (or firm-specific) return at time t can be given as:

$$\epsilon_t = r_t - (\hat{a} + \hat{b} r_{m,t})$$

where $\hat{a}$ and $\hat{b}$ are the estimated values for a and b (market model) and r_t , the observed return of the stock.

Consider the following graph which depicts the cumulative abnormal return of targets following an M&A announcement.

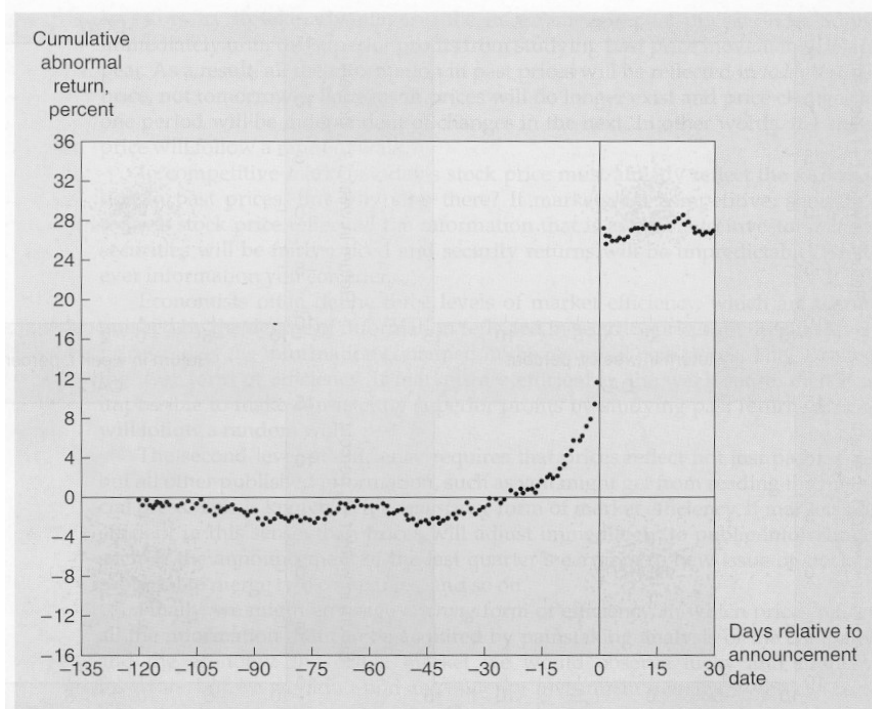


Figure 13.5

The performance of the stocks of target companies compared with that of the market. The prices of target stocks jump up on the announcement day, but from then on, there are no unusual price movements. The announcement of the takeover attempt seems to be fully reflected in the stock price on the announcement day.

Source: A. Keown and J. Pinkerton, “Merger Announcements and Insider Trading Activity,” *Journal of Finance* 36 (September 1981): pp. 855–869.

Overall conclusions:

1. Some studies have shown that most of the stock price response to earnings announcements or dividend payments occur within 10 minutes of the announcement.
2. Stock price reaction to CNBC reports are mostly within 5 minutes of the publication of the report.
3. Many information (i.e. Fed or ECB decisions) are often anticipated and the stock prices reflect these expectations before the announcement.
4. Information is very rapidly reflected in current prices.

7.5 Some anomalies

Over time, some deviations from market efficiency have been observed. They are referred to as “anomalies”.

1. **Small firm** effect: Average annual risk-adjusted returns of portfolios of small stocks are significantly higher than for large stocks.
2. **January** effect: Returns in January are higher than in other months, especially for small stocks.
3. **Book-to-market ratio** effect: Stocks with a high book-to-market ratio exhibit a higher return than those with low ratio.
4. **Monday** effect: Stocks traded on Monday seem to have a lower return than those traded on other days of the week.

Note: These anomalies have been documented and many explanations have been provided to justify their existence.²

Question: Can these anomalies be exploited to generate excess returns? In other words, are there strategies which can be used to exploit these anomalies?

One way to answer these questions is to examine the performance of portfolio managers.

Comparison of the performance of active fund managers and the S&P 500.

²See Bodie, Kane & Marcus (2014) for a detailed discussion.

Active fund managers trail the S&P 500 for the ninth year in a row in triumph for indexing

PUBLISHED FRI, MAR 15 2019 7:09 AM EDT UPDATED FRI, MAR 15 2019 12:52 PM EDT

Bob Pisani@BOBPISANI

KEY POINTS

- Active managers who claim that they would do better during periods of heightened volatility are going to have to find another argument.
- For the ninth consecutive year, the majority (64.49 percent) of large-cap funds lagged the S&P 500 last year.
- After 10 years, 85 percent of large cap funds underperformed the S&P 500, and after 15 years, nearly 92 percent are trailing the index.

It's the triumph of indexing: Fund managers continue to trail their benchmarks.

Active managers who claim that they would do better during periods of heightened volatility are going to have to find another argument.

This week, [S&P](#) Dow Jones Indices released its annual report on how actively managed funds performed against their benchmarks. The conclusion is that active managers continue to show dismal performance against their passive benchmarks. For the ninth consecutive year, the majority (64.49 percent) of large-cap funds lagged the S&P 500 last year.

% of large-cap funds that underperformed S&P 500



Source: S&P Dow Jones Indices • [Get the data](#) • Created with [Datawrapper](#)

"The figures highlight that heightened market volatility does not necessarily result in better relative performance for active investing," the report said.

"What's different about 2018 was the fourth quarter volatility," Aye M. Soe, a managing director at S&P and one of the authors of the report, told CNBC. "Active managers claimed that they would outperform during volatility, and it didn't happen."

The study will bolster the claims of many financial advisors, who say that investing in low-cost, passive funds remains the soundest long-term investment.

This is not a one-year phenomenon. S&P has been doing this study for 16 years, and the long-term results only strengthen the claims for index investing. Indeed, while a fund manager may outperform for a year or two, the outperformance does not persist. After 10 years, 85 percent of large cap funds underperformed the S&P 500, and after 15 years, nearly 92 percent are trailing the index.

Fund managers: long-term underperformance

■ % of fund managers who underperformed their benchmarks (last 10 years)

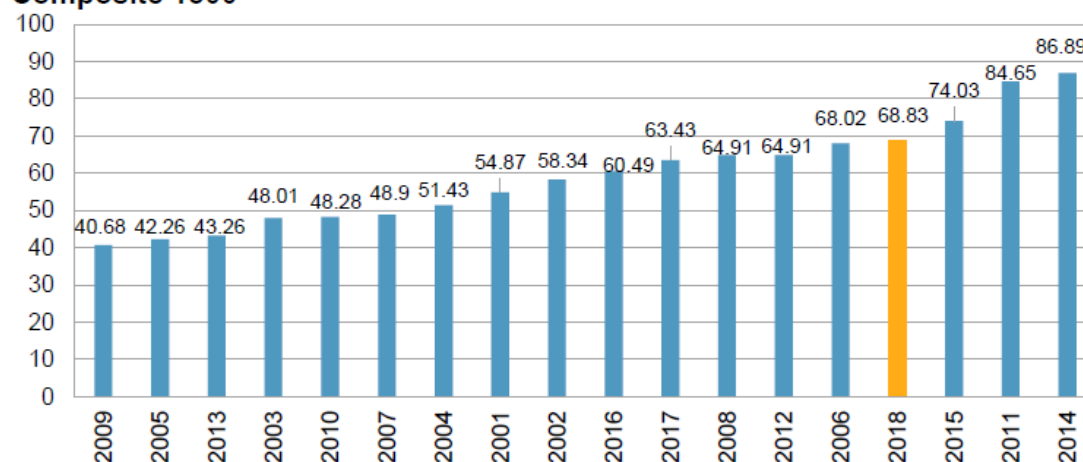


Source: S&P Dow Jones Indices • [Get the data](#) • Created with [Datawrapper](#)

Long-term, the numbers were not much better in other categories like small-cap stocks or fixed income: “Over long-term horizons, 80 percent or more of active managers across all categories underperformed their respective benchmarks,” the report concluded.

Looking at managers’ overall record last year versus the broader S&P 1500 Composite, 2018 was the fourth-worst year for stock managers since 2001.

Exhibit 1: Percentage of Domestic Equity Funds Underperforming the S&P Composite 1500



Source: S&P Dow Jones Indices LLC. Data as of Dec. 31, 2018. Past performance is no guarantee of future results. Chart is provided for illustrative purposes.

Critically, the study adjusts for “survivorship bias.” Many funds are liquidated because of poor performance, so the survivors give the appearance the overall group is doing better than it really is.

“The disappearance of funds remains meaningful,” the report notes. Over 15 years, 57 percent of domestic equity funds and 52 percent of all fixed income funds were merged or liquidated.

The following graph compares the return of actively managed equity funds and the Wilshire 5000 (market-capitalization-weighted index of the market value of all stocks actively traded in the US).

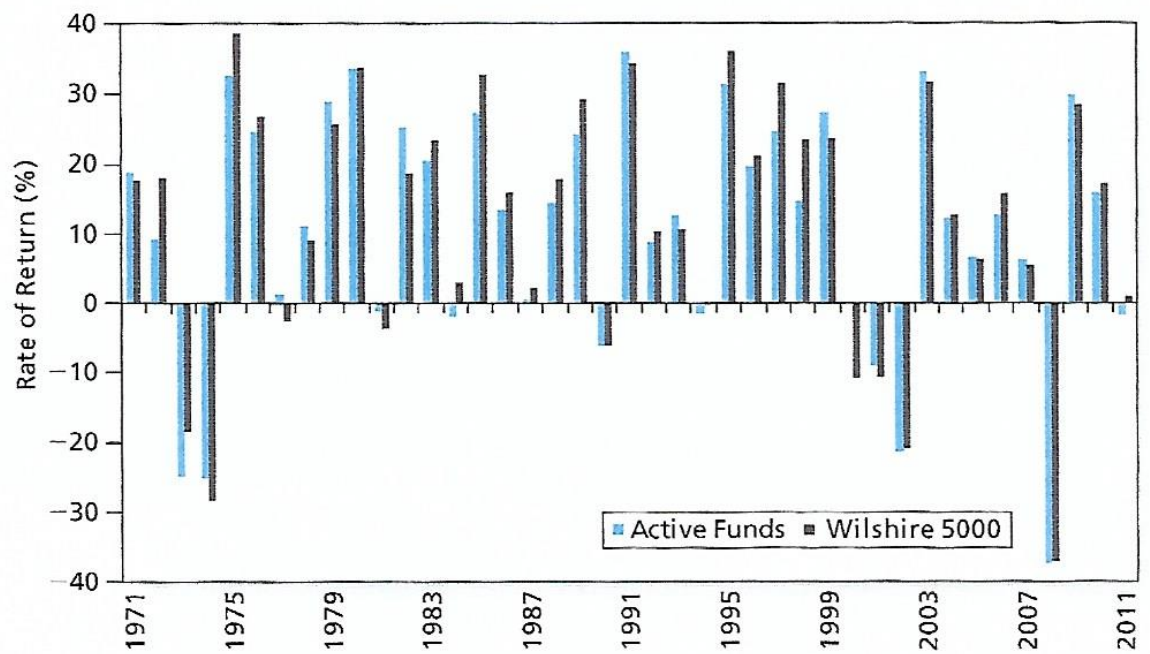


Figure 4.4 Rates of return on actively managed equity funds versus Wilshire 5000 index, 1971–2011

Source: For Wilshire returns, see www.wilshire.com. Used with permission of Wilshire Associates. For active fund returns, see http://www.fiscalisadvisory.com/assets/pdfs/spiva_report_year_end_2011.pdf.

Conclusions:

1. The average return of active diversified equity funds is below the Wilshire 5000 in 25 years over the 41 year period from 1971 to 2011.
2. The average return of the Wilshire index is 1% higher than the average managed equity fund.
3. Over a long period, a passive buy and hold strategy of a large index outperforms an actively managed portfolio by professional managers.

Some quotes:

Active management does not generate superior performance, especially after transaction costs and fees. This fact is surprising. Professionals in almost any field do better than amateurs.

Burton Malkiel, Professor of Finance, Princeton University³

Predictions tell you a lot about those who make them but nothing on the future.

Warren Buffet

Over the past decade, I have attempted to exploit many of the seemingly most promising inefficiencies by actually trading significant amounts of money according to a trading rule suggested by the inefficiencies...I have never yet found one that worked in practice, in the sense that it returned more after cost than a buy-and-hold strategy.

Richard Roll, Professor of Finance, UCLA

I have never known one successful technician, but I have seen the wrecks of several unsuccessful ones? If you commit the social error of asking him why he is broke, he will tell you quite ingeniously that he made the all-too-human error of not believing his own charts. To my great embarrassment, I once chocked at a dinner table of a chartist friend of mine when he made that comment. I have since made it a rule never to eat with a chartist. It's bad for digestion.

Burton Malkiel, Professor of Finance, Princeton University

When Joseph Granville, probably the best known and most followed chartist of the early 1980's was asked how his fool-proof system had led him to make egregious errors in the 1970's, he answered calmly that he was on drugs? When asked why he didn't simply use his system to play the market for himself and thereby make a fortune, he exclaimed that his mission in life was to enrich others, not himself...

Burton Malkiel, Professor of Finance, Princeton University

³See Malkiel (1999), *A Random Walk Down Wall Street*, Norton Press

The technicians (chartists) do not help produce yachts for the customers, but they do help generate the trading that provides yachts for the brokers. Until the public catches on to this bit of trickery, technicians will continue to flourish...

Burton Malkiel, Professor of Finance, Princeton University

I am no longer an advocate of elaborate technique of security analysis in order to find superior value opportunities...Today, I doubt whether successful extensive efforts will generate sufficiently superior selections to justify their costs...I'm on the side of the efficient market school of thought.

Benjamin Graham, father of the fundamental security analysis (in Malkiel (1999))

Question: Does that mean that portfolio managers have no role? Indeed, if capital markets are efficient (weak and semi-strong), is there any purpose in trying to select stocks? Any random choice could do the job?

Some elements of response.

1. As we have seen, portfolio selection rest on a fundamental idea: **diversification**.
2. Therefore, even if capital markets are efficient, any investor should opt for a well-diversified portfolio that will allow for the elimination of non-systematic risk.
3. There are different ways to achieve this objective:
 - Indexed funds or ETFs
 - Mutual funds
4. In an indexed fund or an ETF, investors do not choose their stocks.
5. In a mutual fund, the manager determine the composition of the fund by picking the stocks. Therefore, the choice of stocks may depend on many considerations:
 - Objective of the fund: small or large caps, dividend paying stocks, national or international stocks, industry specific, value or growth stocks, ...
 - Tax considerations for investors. High tax brackets investors may prefer stocks offering potential capital gains to dividend.
 - Risk profile of the investor. Risk aversion varies from one investor to the other and some may prefer portfolios designed to match their risk profile.
 - Individual characteristics of investors such as age and employment. Older investors require a different type of portfolio than younger investor. Their risk profile is different and their investment horizon is different. In addition, investors should minimize investing in stocks of their own employer. For instance, 30% of the shares of Lehman Brothers were owned by their employees at the time of bankruptcy. They not only lost their job but also a total of about \$ 10 billion in share value.

Conclusion: Portfolio managers do have a role even in the presence of efficient capital markets. Their role is not to *beat the market* but to construct portfolios that match the needs of different investors.

Some additional readings.

In Defense of Active Investment

Charles Ellis
CFA
Chairman, Whitehead Institute

In a recent article in the Financial Analyst Journal (Vol. 71, no. 4, July-August 2015), Charles Ellis, an experienced portfolio manager wrote the following:

Nobody doubts that efficient markets are a major social good in many ways. By enabling outsiders to participate with confidence, knowing that security prices are fair for both buyers and sellers and that transaction costs are low, efficient markets encourage millions of investors to trust the capital markets with their savings. Efficient markets thus enable growing companies to raise substantial amount of capital at low cost. They also enable stronger companies to acquire weaker companies and to redeploy assets in ways that are more socially productive....

Although my reverie could be easily dismissed as mere fantasy, my observations over many years have convinced me that very few investment managers are able to match or beat the market, particularly after adjusting for risk and after deducting costs and fees. Fortunately, all investors have access, through low-cost indexing, to funds that match the market returns and no more than the market level of risk. And this access enables investors to concentrate on the winners' game of defining their own unique objectives and designing long-term investment portfolios that are most likely to meet those objectives.

The hypothesis of capital market efficiency has attracted a great deal of interest and critical comment. This is somewhat surprising because capital market efficiency is a fairly limited concept. It says that the prices of securities instantaneously and fully reflect all available relevant information. It does not imply that product markets are perfectly competitive or that information is costless.

Capital market efficiency relies on the ability of arbitrageurs to recognize that prices are out of line and to make a profit by driving them back to an equilibrium value consistent with available information. Given this type of behavioral paradigm, one often hears the following questions: If capital market efficiency implies that no one can beat the market (i.e., make an abnormal profit), then how can analysts be expected to exist since they, too, cannot beat the market? If capital markets are efficient, how can we explain the existence of a multibillion dollar security analysis industry? The answer, of course, is that neither of these questions is inconsistent with efficient capital markets. First, analysts can and do make profits. However, they compete with each other to do so. If the profit to analysis becomes abnormally large, then new individuals will enter the analysis business until, on average, the return from analysis equals the cost (which, by the way, includes a fair return to the resources that are employed). As shown by Cornell and Roll [1981], it is reasonable to have efficient markets where people earn different gross rates of return because they pay differing costs for information. However, net of costs their abnormal rates of return will be equal (to zero).

As we shall see in the next chapter, the concept of capital market efficiency is important in a wide range of applied topics, such as accounting information, new issues of securities, and portfolio performance measurement. By and large the evidence seems to indicate that capital markets are efficient in the weak and semistrong forms but not in the strong form.

A funny and revealing experiment on price predictions.

Value	Price 12/09/2011	Price objective 12/12/2011	Potential return 3 months	Recommendation 12/09/2011	Price 12/10/2011	Variation %	Price 12/11/2011	Variation %	Price 01/12/2011	Variation %	Diff
SG	15,67	42	168,0%	Reinforce	23,2	48,05%	18,9	20,61%	17,6	12,32%	-155,71%
BNP Paribas	26,12	62	137,4%	Buy	35,4	35,53%	32,2	23,28%	28,7	9,88%	-127,49%
Crédit Agricole	4,83	9,7	100,8%	Constant - Reinforce	5,6	15,94%	4,9	1,45%	4,6	-4,76%	-105,59%
Natixis	2,26	3,9	72,6%	Constant - Reinforce	2,5	10,62%	2,2	-2,65%	2,1	-7,08%	-79,65%
Carrefour	15,26	20	31,1%	Constant	17,9	17,30%	20,2	32,37%	19,8	29,75%	-1,31%
Vivendi	15,46	21,8	41,0%	Reinforce	16,7	8,02%	15,9	2,85%	17	9,96%	-31,05%
Axa	8,48	16,9	99,3%	Reinforce	11,8	39,15%	10,6	25,00%	10,6	25,00%	-74,29%

Data: Boursorama

Value	Price 02/04/2012	Price objective 02/07/2012	Potential return 3 months	Recommendation 02/04/2012	Price 14/05/2012	Variation %	Price 24/05/2012	Variation %	Price 11/06/2012	Variation %	Diff
SG	21,24	27,00	27,0%	Reinforce	16,50	-22,32%	16,6	-21,85%	18,28	-13,94%	-40,94%
BNP Paribas	34,66	45,00	29,8%	Reinforce	27,62	-20,31%	27,03	-22,01%	29,47	-14,97%	-44,80%
Crédit Agricole	4,52	5,10	12,8%	Keep	3,27	-27,65%	3,06	-32,30%	3,30	-26,99%	-39,82%
Natixis	2,84	3,13	10,1%	Reinforce	2,03	-28,52%	1,99	-29,93%	2,10	-26,06%	-36,17%
Carrefour	17,74	17,75	0,08%	Decrease	14,00	-21,08%	14,67	-17,31%	14,38	-18,94%	-19,02%
Vivendi	13,72	16,50	20,3%	Reinforce	12,43	-9,40%	13,07	-4,74%	13,78	0,44%	-19,82%
Axa	12,25	16,00	30,6%	Reinforce	9,51	-22,37%	9,54	-22,12%	10,60	-13,47%	-44,08%

Data: Boursorama

éditorial d'opinion

31/10/05 Le Tribune

« Greed » et « fear », les deux mamelles du marché

Appât du gain et la peur sont les deux moteurs des marchés financiers. Les spéculateurs professionnels jouent à merveille de ces deux cordes, qui engendrent une volatilité inégale. Et une accélération du rythme des crises financières. Le marché en apparaît de plus en plus humain, d'où la folie qui s'empare de lui, parfois, comme aujourd'hui.



À CONTRE-COURANT
MARC FIORENTINO
Stratège d'Allofinance.com

Dire que les marchés sont volatils depuis quelques semaines est un doux euphémisme. Ils ont une fois de plus perdu la raison, si l'on peut penser qu'ils l'aient seulement eue un jour. Le début de la semaine qui vient de s'écouler semblait sonner le glas de tous les accitifs « à risque ». On avait jeté depuis longtemps le bébé avec le bain mais on se débarrassait même des tétines et des couches qui traînaient encore. Il ne fallait plus rien détenir. Rien. Tout actif autre qu'une obligation allemande

était hautement radioactif. Et quand des personnes extérieures au monde de la finance me demandaient ce qu'il se passait, je me suis vite trouvé à court d'arguments. Comment expliquer la folie de façon raisonnable ? J'ai bien tenté d'évoquer les craintes de récession en Europe (alors que l'OCDE relevait ses prévisions de croissance...) ou aux États-Unis (alors que les indicateurs économiques indiquaient une reprise forte, notamment dans l'immobilier), les craintes d'explosion de la bulle chinoise (oui,

mais pourquoi aujourd'hui plus qu'hier ?) ou le risque de désintégration de l'euro (risque auquel je ne crois pourtant pas une seule seconde). Mais j'avais du mal à convaincre, n'étant moi-même pas convaincu. J'ai honte, mais je dois même avouer qu'à court d'arguments, j'ai évoqué la tension en Corée du Nord... C'est pitoyable, non ? Il fallait vraiment que je n'aie plus rien à dire...

Alors, pourquoi cette panique avec un CAC à 3.287 points le 25 mai ? Mais pourquoi aussi l'euphorie le 15 avril avec un CAC proche des 4.100 points ? Soyons honnêtes. Il n'y a pas d'explication rationnelle, ni dans un cas, ni dans l'autre. Car les marchés sont en fait guidés, et uniquement guidés par « greed », l'appât du gain, et « fear », la peur.

Les professionnels de la spéculation le savent et jouent parfaitement de ces deux cordes extrêmement sensibles. Pour qu'un « initié » gagne de largent, il faut parvenir soit à galvaniser la foule, soit à la faire paniquer. Il faut convain-

cre, comme en 2000, les petits porteurs que les arbres de l'internet, même si ce sont encore de jeunes pousses brûlant des fonds par millions, ne peuvent que monter vers le ciel et que c'est la dernière chance de prendre le train en marche (greed), comme il faut les persuader (en mai 2010) que l'Europe est en faillite, que l'euro va disparaître et qu'ils n'ont plus que quelques heures pour se jeter dans la mer pour tenter de rejoindre à la nage les côtes américaines, pourtant polluées par la marée noire (fear).

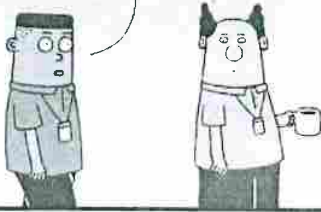
Les marchés ne sont pas efficients. Ils ne l'ont jamais été. Mais ils le sont encore moins aujourd'hui que par le passé. La preuve ? Le rythme des crises financières. Sur les vingt-cinq dernières années, le rythme des crises est passé de trois ans à dix-huit mois. Une crise tous les dix-huit mois et des crises d'une ampleur de plus en plus grande ! Une bulle, une crise, une bulle, une crise, une bulle, une crise, et ainsi de suite. Greed, puis fear. C'est une remise en question complète de tous les

modèles classiques d'analyse et de prévision à laquelle nous assistons.

Aucun « fondamentaliste » ne pourra expliquer le CAC à 4.085 points, puis le CAC à 3.287 points quelques jours après, et peut-être un CAC au-dessus de 4.000 points avant la fin de l'année. À l'ère où les ordinateurs et le « high speed trading » règnent en maîtres sur les marchés, le paradoxe, c'est que les marchés deviennent de plus en plus « humains ». Les termes qu'on utilise d'ailleurs pour les décrire relèvent de plus en plus de la psychologie, et parfois même de la psychiatrie... L'homme, surtout les hommes, d'ailleurs, ont toujours été guidés par l'appât du gain et par la peur (par le sexe, aussi, mais cela n'a rien à voir avec notre sujet...). Les grands fonds spéculatifs l'ont compris depuis longtemps et exploitent parfaitement cette faiblesse. Alors, deux conseils : ne cherchez pas d'explication à ce qui est inexplicable, et ne tombez pas dans le piège que le diable vous tend... Prenez de la hauteur... ■

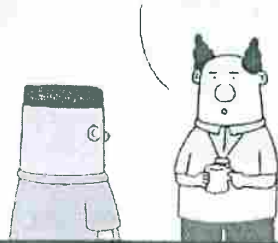
1

I FOLLOWED YOUR
INVESTMENT ADVICE
AND LOST ALL OF
MY SAVINGS IN THE
STOCK MARKET.



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DID I MENTION THAT
PAST PERFORMANCE
IS NOT AN INDICATION
OF FUTURE RETURNS?



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THEN ...
HOW DOES
"ADVICE"
ACTUALLY
WORK?

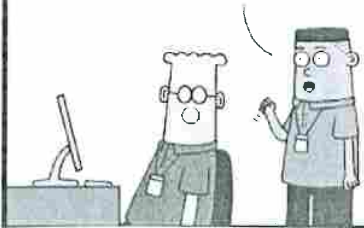


IT ONLY
WORKS
FOR THE
PEOPLE
THAT
GIVE IT.



2

I BOUGHT MY FIRST
STOCK AND IT WENT
UP FIVE PERCENT IN
ONE WEEK!



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THAT MEANS I'M A
STOCK-PICKING
GENIUS. I PLAN TO
MAX OUT ALL OF MY
CREDIT CARDS AND
BECOME A DAY-
TRADER.

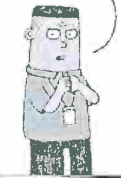


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THE
TOTAL
MARKET
WAS UP
SIX
PERCENT.

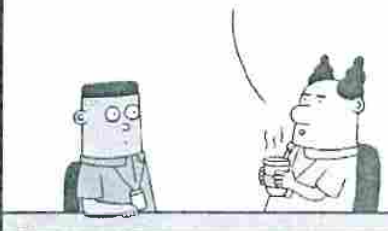


THAT'S
JUST
LUCK. IT
CAN'T DO
THAT
FOREVER.



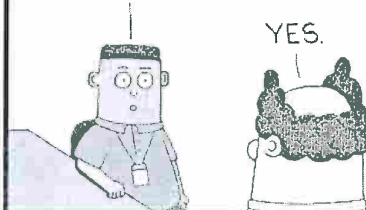
3

ASOK, YOU CAN BEAT
MARKET AVERAGES
BY DOING YOUR OWN
STOCK RESEARCH.



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SO ... YOU BELIEVE
EVERY INVESTOR CAN
BEAT THE AVERAGE BY
READING THE SAME
INFORMATION?



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MAKES YOU WONDER
WHY MORE PEOPLE
DON'T DO IT.



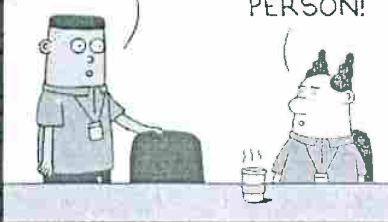
JUST LAZY,
I GUESS.



4

DO YOU
HAVE ANY
INVESTMENT
TIPS?

YOU'RE
ASKING
THE RIGHT
PERSON!



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I CAN TEACH YOU
HOW TO TIME THE
MARKET, CATCH A
FALLING KNIFE, AND
INVEST IN A DEAD-
CAT BOUNCE. THAT'S
MY SYSTEM.



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WHAT
ABOUT
DIVERSI-
FICATION?



I DON'T
INVEST
IN ANY-
THING I
CAN'T
SPELL.



5

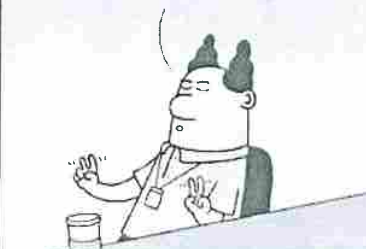
LESSONS IN INVESTING

YOU SHOULD BUY A
STOCK WHENEVER THE
CHART LOOKS LIKE A
SQUIRREL SITTING ON
A CLOWN'S SHOULDER.



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THAT'S CALLED
"TECHNICAL
ANALYSIS."



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I'M NOT
GOING TO
DO THAT.



GOOD,
BECAUSE
IT DOESN'T
WORK IF
EVERYONE
DOES IT.



The statistical anomaly that i

Strategies

JEFF SOMMER

Warren Buffett is probably the most famous investor of his generation, and for good reason: His track record over the long term is a thing of beauty.

He has beaten the market by a wide margin over 49 years, a record so impressive that it's used in finance classes as a textbook example of "alpha."

Alpha is an elusive quality. Very simply put, it is the ability to beat an index fund without adding risk to a portfolio. Investment managers are always seeking it. If it exists, Warren Buffett surely has had it.

A new statistical analysis of Mr. Buffett's long-term record at Berkshire Hathaway has just been done, and it has come up with some fascinating insights about his abilities, past and present, and about the chances that the rest of us have for beating the market. Using a series of statistical measures, the study suggests that Mr. Buffett has indeed been blessed with an impressively big dose of alpha over a very long career.

But it also reveals something that isn't impressive at all: For four of the last five years, Mr. Buffett has been doing worse than the typical, no-frills Standard & Poor's 500-stock index fund — so much worse that it's unlikely to be a matter of a string of bad luck. Mr. Buffett has begun to behave like an investor with no alpha at all.

Both sobering facts — Mr. Buffett's long-term outperformance and his recent stretch of mediocrity — appear in high relief in the analysis conducted by Salil Mehta, an independent statistician with deep experience in Washington and on Wall Street. Part of the study appears on his blog, Statistical Ideas, and he shared the rest of it with me, in an elaborate spreadsheet filled with more than 30 pages of data and formulas.

Mr. Mehta, who served as director of analytics in the Treasury Department for the \$700 billion Troubled Asset Relief Program, and as director of policy, research and analysis for the Pension Benefit Guaranty Corporation, says Mr. Buffett's record provides some humbling lessons about investment strategies.

"It shows how amazingly difficult it is to keep beating the market, even for a master like Warren Buffett," Mr. Mehta said in an interview. "And it suggests



From the beginning of 1985 through the end of 2013, Warren Buffett outperformed the

that just about everybody else should just use index funds and not even think about trying to beat the market."

Mr. Buffett draws the same conclusion about index funds in his recent annual letter to shareholders of Berkshire Hathaway, of which he is chairman. At 83, he writes that he has given the following explicit instructions for the money that he is bequeathing in a trust for his wife: "Put 10 percent of the cash in short-term government bonds and 90 percent in a very low-cost S.&P. 500 index fund. (I suggest Vanguard's.) I believe the trust's long-term results from this policy will be superior to those attained by most investors — whether pension funds, institutions or individu-

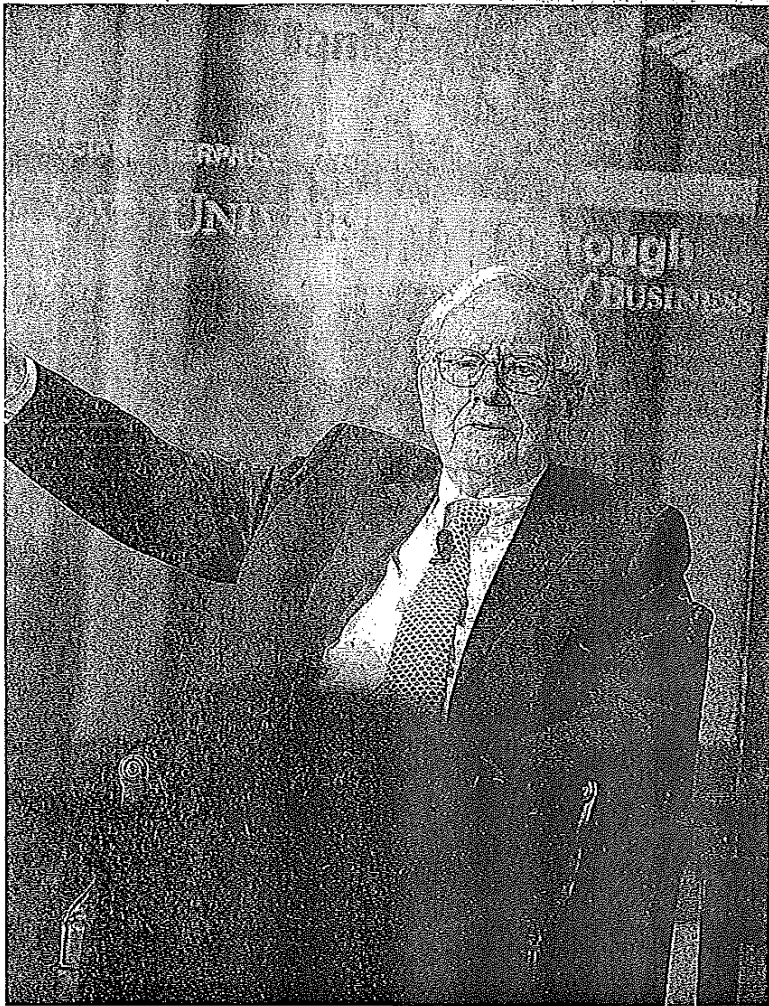
als — who employ high-fee managers

Mr. Buffett says he has amassed his long-term record by following the precepts of Benjamin Graham, the value investor who was his mentor and professor at Columbia University. Like Mr. Graham, Mr. Buffett says he tries to make investments that will last a lifetime.

Among the forms of active management that Mr. Buffett warns about, he includes high-frequency trading, which is the subject of the new book, "Flash Boys: A Wall Street Revolt," by Michael Lewis. (An excerpt appears in The New York Times Magazine.) But he also says he's prepared to exploit some of the problems induced by "flash traders."

ECONOMY COMPANIES BUSINESS

s Buffett, the alpha investor



ANDREW HARRER/BLOOMBERG NEWS

P. 500-stock index, including dividends, at an annualized pace of 9.9 percentage points.

A “flash crash” or another extreme drop in the market won’t hurt someone who is a “true investor if he has cash available when prices get far out of line with values,” Mr. Buffett says in the letter. “A climate of fear is your friend when investing; a euphoric world is your enemy.”

Such precepts, however, don’t really explain how Mr. Buffett outperformed the market; if they did, anyone who had read his annual letters could have done it. Mr. Mehta doesn’t explain how he did it, either. Instead, Mr. Mehta points out how unusual Mr. Buffett really is. From a statistical standpoint, he is an anomaly.

Consider this, gleaned from Mr. Buf-

fett’s own data on the second page of the Berkshire annual report: From the beginning of 1965 through the end of 2013, he outperformed his own chosen benchmark — the S.&P. 500-stock index, including dividends — by 9.9 percentage points, annualized.

How rare is that? According to Mr. Mehta’s analysis, it puts Mr. Buffett in a vanishingly small class, comprising far less than 1 percent of the population of investors. This is the tiny group that is statistically likely to have been able to beat the stock market through that elusive alpha — skill of some sort, rather than just chance — over a period of 45 to 50 years.

The flip side of Mr. Mehta’s finding is

also worth considering. A vast majority of individuals, including most people now working in finance, do not have alpha, Mr. Mehta says. It doesn’t matter whether they have studied finance or have prodigious math skills; the statistics show that they are unlikely to have the ability to beat the market.

That has a serious implication for individual investors, he says: True investing skill is so rare that the rest of us shouldn’t even try to emulate those who have it. In addition, he says, we probably shouldn’t bother trying to hire the few outperformers to invest our money. Why? Because we aren’t likely to be able to identify them. Their talents aren’t always on public display, and there may be only a few thousand of them in the entire United States.

Mr. Buffett’s talents are widely known. But despite his celebrated past performance, his returns since the beginning of 2009 have been disappointing.

In four of the last five calendar years, he has underperformed his own benchmark, the S.&P. 500 with dividends, often by significant margins. (In 2011, his return of 4.6 percent beat the benchmark by 2.6 percentage points. In addition, data provided by Morningstar shows that he underperformed the average stock mutual fund investor in four of the five years.)

By contrast, in the previous decades, he had underperformed the S.&P. only six times. Mr. Mehta said his calculations showed that given such a long period of outperformance, there is only a 3 percent chance that the recent stretch of underperformance was a matter of bad luck.

What happened? We don’t really know, and Mr. Buffett declined to comment for this column. In a section of the annual report, Mr. Buffett notes that because he judges investments based on what he considers to be their intrinsic value, Berkshire may well underperform in periods when stocks rise rapidly. He also says that as Berkshire has grown, he has increasingly been buying entire operating companies, as opposed to investing in shares of publicly traded companies, and, as a result, the long-term strength of his investments may not be fully reflected in his annual data.

Will Mr. Buffett return to form and trounce the market again? He might or he might not.

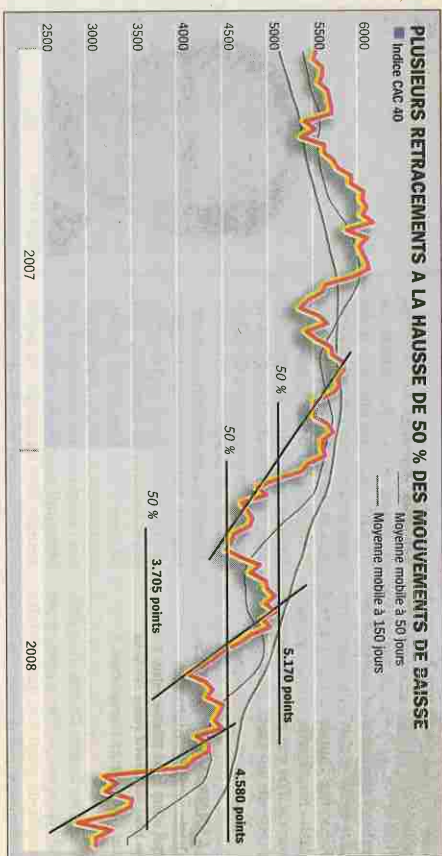
But Mr. Mehta says that most people will be better off with a draw: Anyone can match the market with an index fund.

Approche Fibonacci

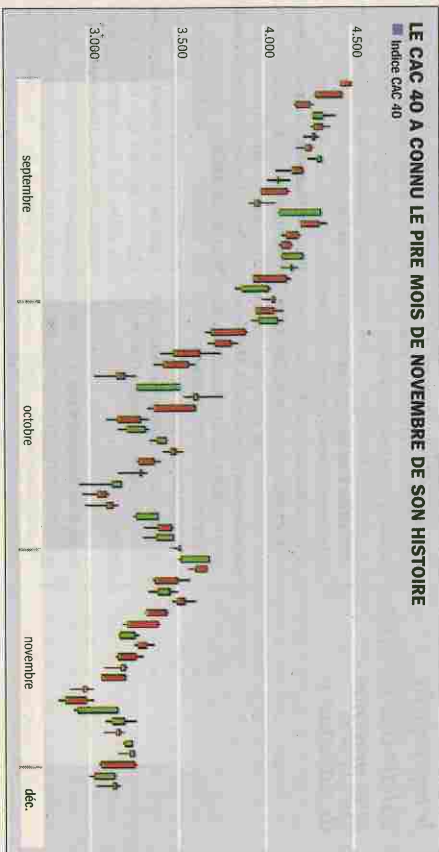
Les grands mouvements ont cela de particulier qu'ils mettent à mal tous les repères techniques classiques. Fondée sur le nombre d'or, la théorie de Fibonacci part de l'idée que le marché corrige de 31,8 %, 50 % ou 61,8 % le mouvement qui a précédé. Mais, lors de l'écroulement de la bulle Internet, en septembre 2000, et lors de la correction actuelle entrecisée en juillet 2007, cette théorie a été battue en brèche. Le marché a ainsi fait voler en éclats ses trois retracements de Fibonacci, à 4,740 points (38,2 %), 4,300 points (50 %) et 3,200 points (61,8 %).

Sur le court terme, en revanche, le retracement de 50 % s'est révélé être un support très pertinent. Ainsi,

depuis octobre 2007, l'indice CAC 40 reproduit à l'identique le même schéma. Une vague de baisse puis saine s'étalant sur une période de deux mois et demi à trois mois et demi, suivie d'un rebond qui retracer la moitié (50 %) du dernier mouvement, sans pouvoir aller au-delà. Cela a été le cas lorsque l'indice a buté en mai contre la résistance de 5,170. Cela a de nouveau été le cas début septembre à l'approche de 4,600 points. Cette semaine n'a pas fait exception, avec un CAC 40 qui a buté contre la résistance de 3,280 points, soit la moitié du chemin parcouru lors de la baisse initiée début novembre. Au-dessous, la tendance reste baissière.



LE CAC 40 A CONNU LE PIRE MOIS DE NOVEMBRE DE SON HISTOIRE



Les chandeliers japonais

La hausse historique de 10 % le 24 novembre a laissé comme un goût d'inachevé. Le long chandelier blanc (*vert sur notre graphique*) laissait espérer une accélération dans les jours suivants. Mais la figure en « avalement baissier » dessinée le 1^{er} décembre a mis à mal ce scénario. Le « doji porteur deau » laissé jeudi atteste l'hésitation dans laquelle se trouve désormais le marché. Il manque désormais une figure de continuation forte de type « marubozu », c'est-à-dire un long chandelier blanc (*vert et contre*) sans ombre, pour relancer la dynamique haussière.

L'observation des chandeliers en données hebdomadaires per-

met de se faire une idée plus précise de la tendance à moyen terme. Avec deux longs chandeliers rouges, le CAC 40 a connu la pire quinzaine de son histoire. Après la semaine noire de début octobre, il conforte ainsi une tendance de fond fragile. Il faudrait, pour que la tendance s'inverse, au moins une semaine ou deux de forte hausse, matérialisée par de longs chandeliers verts. Pour l'heure, les hausses hebdomadaires de l'indice ne suffisent pas à compenser les baisses qui précèdent. Seul point positif, la baisse subie cette semaine n'a pas entamé la moitié du chandelier vert dessiné la semaine dernière.

8 Options Valuation

As we have seen in the previous sections, investors can acquire different types of securities on financial markets. These securities are risky since their market price can vary over time. One way to reduce and manage the risk associated to these securities is to use “options”.

8.1 Definitions

An **option** is a contract which grants its owner the right to buy (**call**) or sell (**put**) an underlying security at a given price (strike) at or up to a given date (maturity or expiration date).

The underlying securities can be:

- stocks
- stock market indices
- currencies
- bonds
- gold
- commodities (oil, gas, ...)

There are two important aspects of an option contract:

- It is a **right** to buy or sell but not an obligation.
- The decision to **exercise** or not this right belongs to the owner of the option.

There are two basic type of options:

- **Call** option: right to buy an underlying asset at a predetermined price
- **Put** option: right to sell an underlying asset at a predetermined price

Remarks:

- Options are issued by a national registered clearing corporation.
- US: Options Clearing Corporation (OCC)
- France: Marché d’Options Négociables de Paris (MONEP)

Notation:

- C_0 : price (premium) of a call option at time 0 (initiation of the strategy).

Price paid by the buyer and received by the seller of a call option in exchange for the right given to the buyer of the call to exercise or not the option and buy the underlying asset at maturity.

- P_0 : price (premium) of a put option at time 0 (initiation of the strategy)

Price paid by the buyer and received by the seller of a put option in exchange for the right given to the buyer of the put to exercise or not the option and sell the underlying asset at maturity.

- C_T : price of a call option at time T (maturity).
- P_T : price of a put option at time T (maturity).
- S_T : price of the underlying security at time T (maturity).
- K : exercise price or **strike**: unit price to be paid (received) by the investor to buy (sell) the underlying asset if the option is exercised.
- T : Maturity date of the option. At this date, the option expires and the option becomes worthless.
- Π : Net profit of the strategy per unit of the underlying asset.

Remarks

- Typically, an option contract allows an owner to buy or sell 10 or 100 units of the underlying security.
- If the option is on 100 shares, the total price of an option contract is then equal to $100 \times$ the unit price (premium). Ex: If a call option on Apple trades at a premium of \$3, the total cost of the option is equal to \$300.

Options differ according to the expiration date specified in the contract.

- **European** options can be exercised only at expiration.
- **American** options can be exercised at any time before expiration.

8.2 Value of Options at Maturity

1. Call option

At maturity (T), if the price of the underlying asset is S_T , a **call** option with a strike of K is worth:

$$C_T = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases}$$

Explanation:

- If the market price of the underlying asset, S_T , is greater than the strike, K , the investor exercises its contract and buys the underlying asset at the strike. Indeed, an investor would pay less than the market price for the underlying asset.

Ex: Let $S_T = 110$ and $K = 100$. This means that the owner of the call option can buy an asset worth 110 and pay only 100.

- Payoff (gross profit) per unit of the underlying asset: $S_T - K$
- Net profit per unit: $\Pi = (S_T - K) - C_0$
- Otherwise, the investor does not exercise the contract
 - Payoff (gross profit) per unit of the underlying asset: 0
 - Net profit per unit: $\Pi = -C_0$
 - One should note that an investor who does not exercise his call option can still buy the underlying asset directly on the market. He simply lets the option expire.

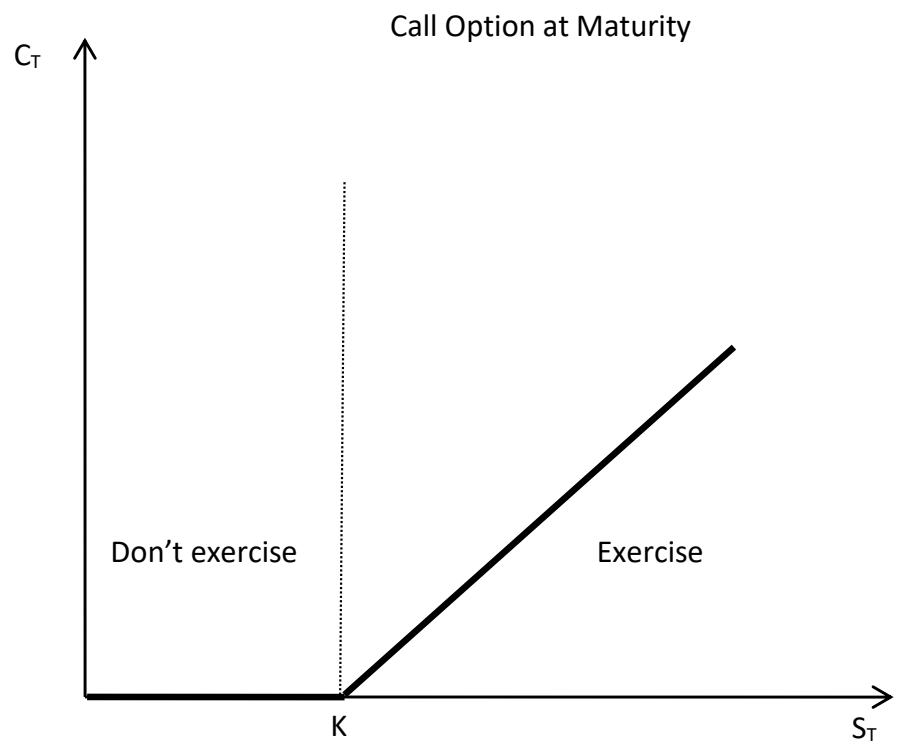
Thus, we can write:

$$C_T = \text{Max } [0, S_T - K]$$

and

$$\Pi = \text{Max } [0, S_T - K] - C_0$$

Remark: The net profit of an option strategy is somehow misleading since it does not take into account the present value of the option's payoff.



2. Put option

At expiration (T), if the price of the underlying asset is S_T , a **put** option with strike price K is worth:

$$P_T = \begin{cases} 0 & \text{if } S_T > K \\ K - S_T & \text{if } S_T \leq K \end{cases}$$

Intuition:

- If the market price of the underlying asset, S_T , is less than the strike, K , the investor exercises its contract and sells the underlying asset at the strike. Indeed, an investor would receive more than the market price for the underlying asset.

Ex: Let $S_T = 90$ and $K = 100$. This means that the owner of the put option can sell an asset worth 90 and receive 100.

- Payoff (gross profit) per unit of the underlying asset: $K - S_T$.
- Net profit per unit: $\Pi = (K - S_T) - P_0$.
- Otherwise, the investor does not exercise the contract.
 - Payoff (gross profit) per unit of the underlying asset: 0
 - Net profit per unit: $\Pi = -P_0$
- One should note that an investor who does not exercise his put option can still sell the underlying asset directly on the market. He simply lets the option expire.

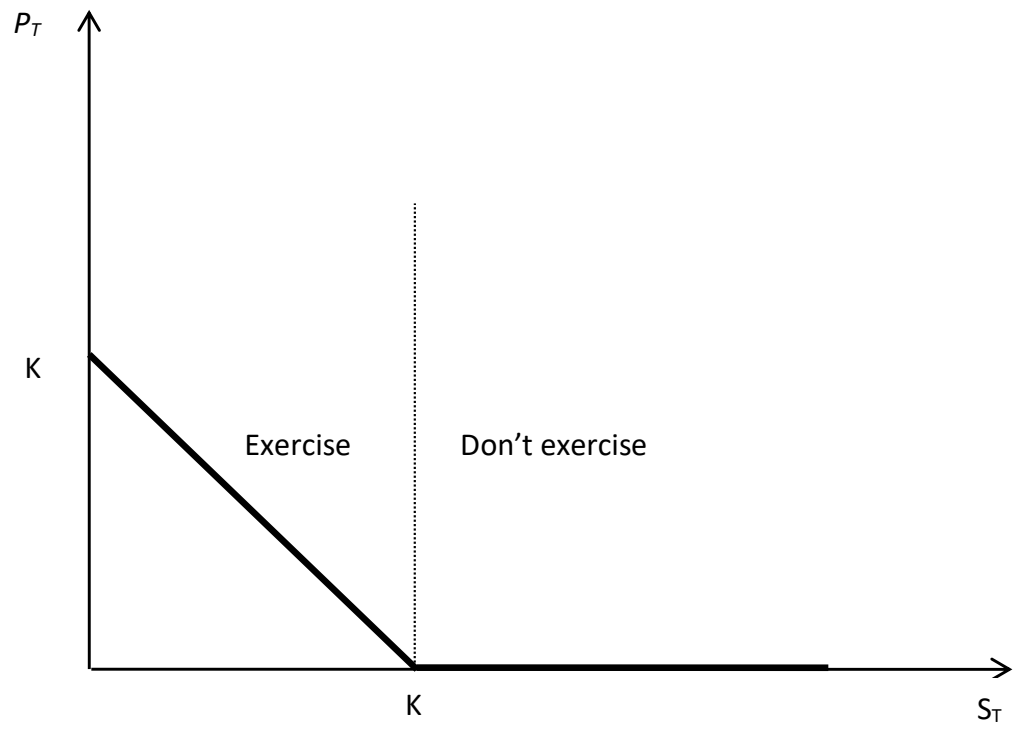
Thus, we can write:

$$P_T = \text{Max } [0, K - S_T]$$

and

$$\Pi = \text{Max } [0, K - S_T] - P_0$$

Put Option at Maturity



Example

1. Call option at maturity

- Calculate the gross payoff and net profit per share from buying a call option on a stock with a strike/exercise price $K = \$100$ and a premium $C_0 = \$10$
- Let's consider different price of the stock at maturity.

S_T	80	90	100	110	120	130
Payoff	0	0	0	10	20	30
Net profit	-10	-10	-10	0	10	20

Explanation:

- If $S_T \leq 100$: the option is not exercised and the investor loses the premium for the call.
- If $S_T > 100$: the option is exercised. The investor loses the premium but makes money on the exercise of the call. He breaks even at $S_T = 110$.

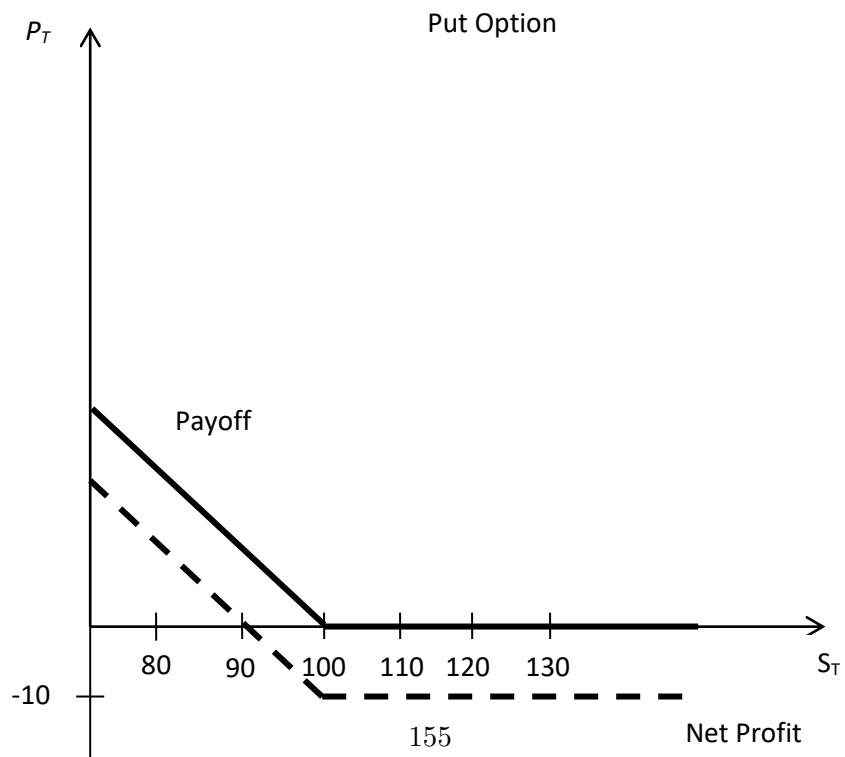
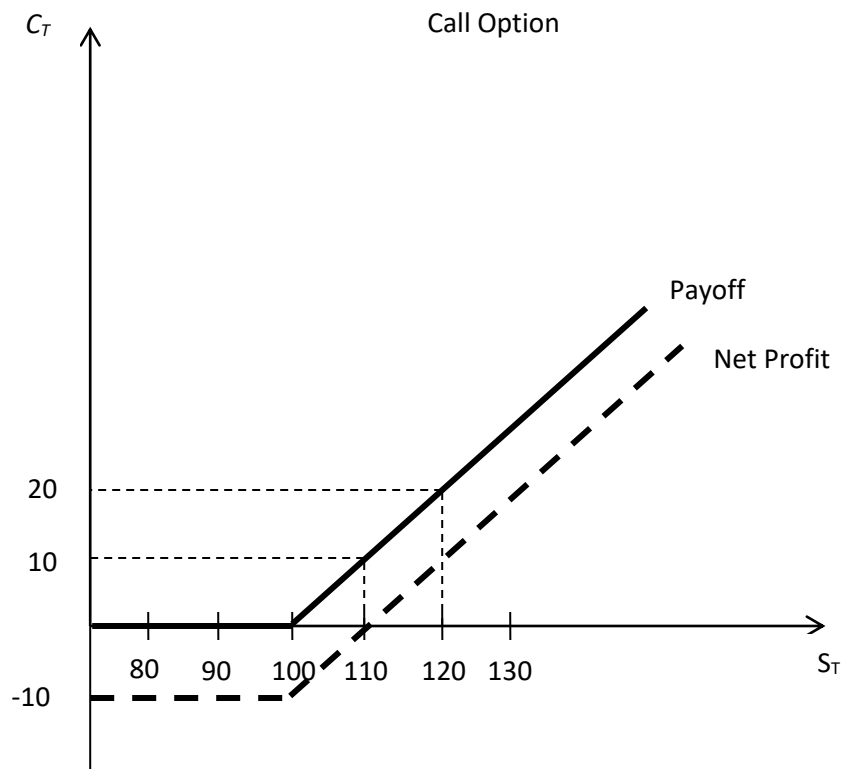
2. Put option at maturity

- Calculate the gross payoff and net profit per share from buying a put option on a stock with a strike/exercise price $K = \$100$ and a premium $P_0 = \$10$
- Let's consider different price of the stock at maturity.

S_T	80	90	100	110	120	130
Payoff	20	10	0	0	0	0
Net profit	10	0	-10	-10	-10	-10

Explanation:

- If $S_T \geq 100$: the option is not exercised and the investor loses the premium for the put.
- If $S_T < 100$: the option is exercised. The investor loses the premium but makes money on the exercise of the put. He breaks even at $S_T = 90$.



8.3 Investment strategies with options

There are four possible positions with plain options.

- Buy a call
- Sell a call
- Buy a put
- Sell a put

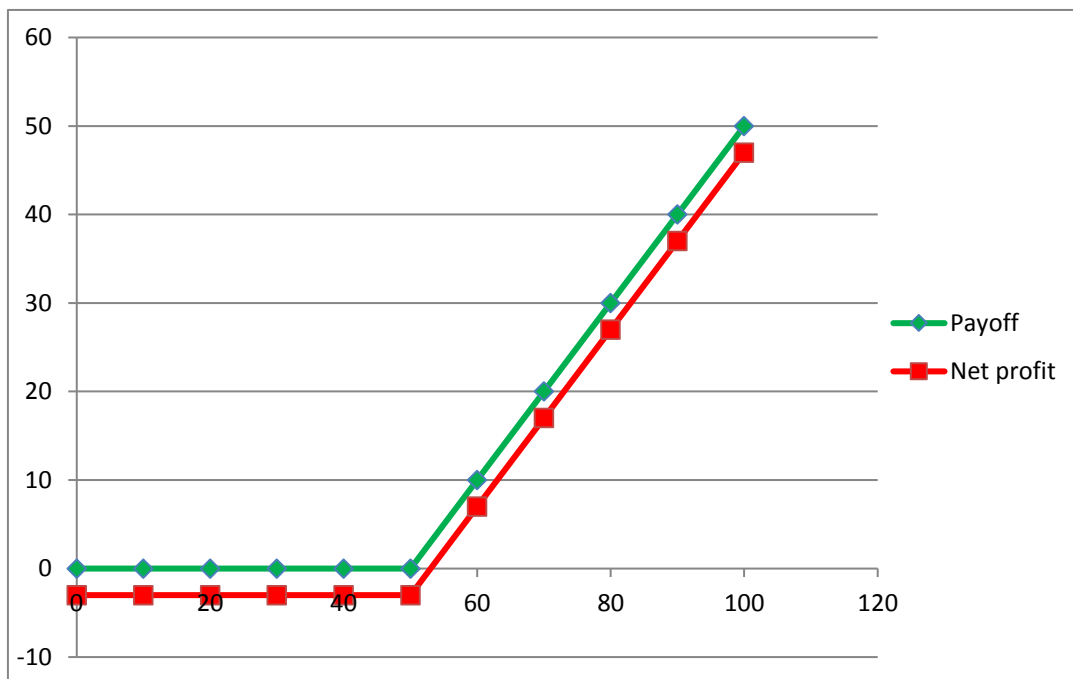
The following graphs illustrate the payoff and net profit for the buyer and the seller of a call and a put respectively.

Remarks:

- Option trading is a zero-sum game. The gains of one party are equal to the losses of the counterparty.
- If the buyer of an option exercise his option, he makes a profit. This is equal to loss for the seller.
- The payoff and net profit functions of the option buyer are the mirror image of the payoff and net profit functions of the option seller.

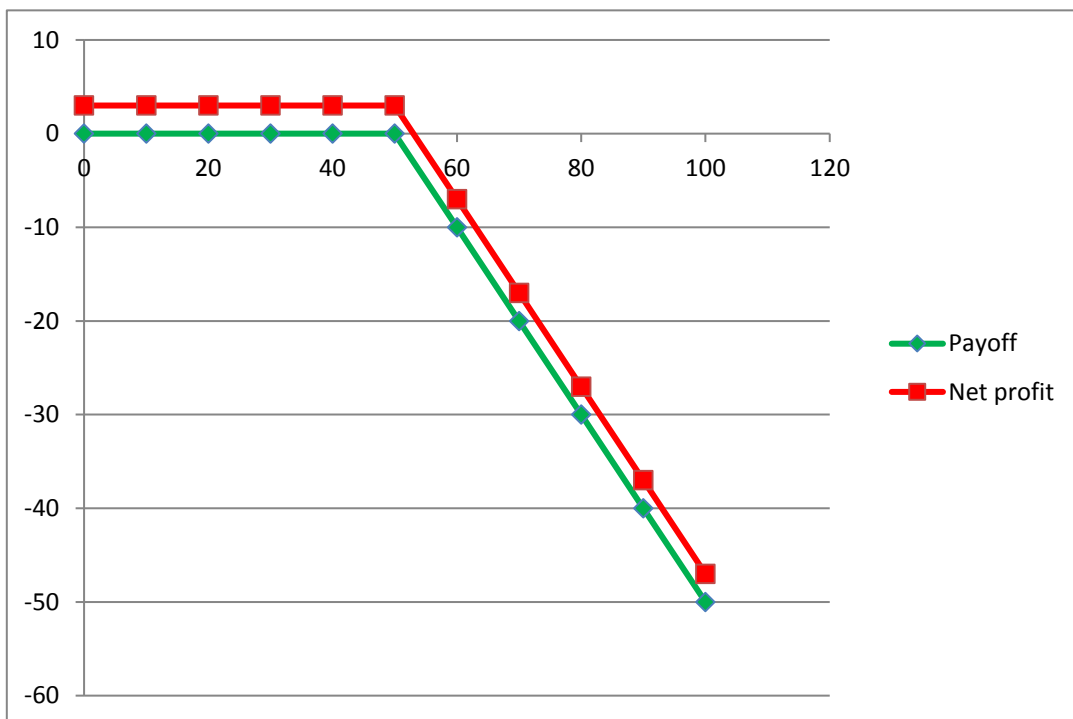
Buy a call

Premium	3										
Strike	50										
S	0	10	20	30	40	50	60	70	80	90	100
Payoff	0	0	0	0	0	0	10	20	30	40	50
Net profit	-3	-3	-3	-3	-3	-3	7	17	27	37	47



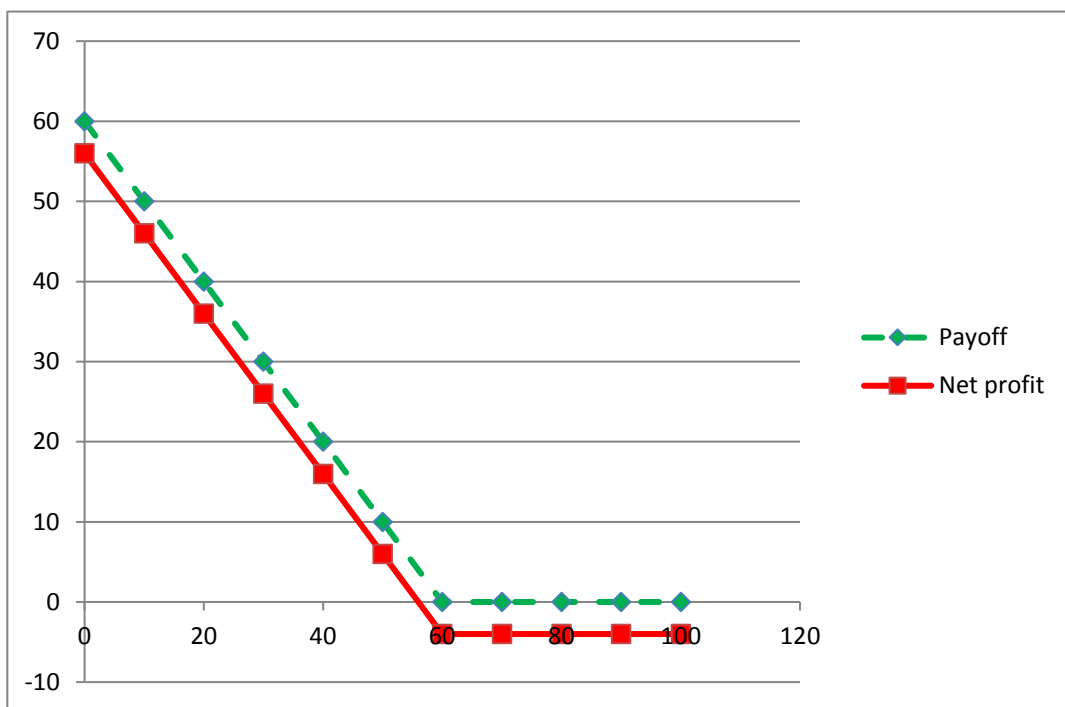
Sell Call

Premium	3										
Strike	50										
S	0	10	20	30	40	50	60	70	80	90	100
Payoff	0	0	0	0	0	0	-10	-20	-30	-40	-50
Net profit	3	3	3	3	3	3	-7	-17	-27	-37	-47



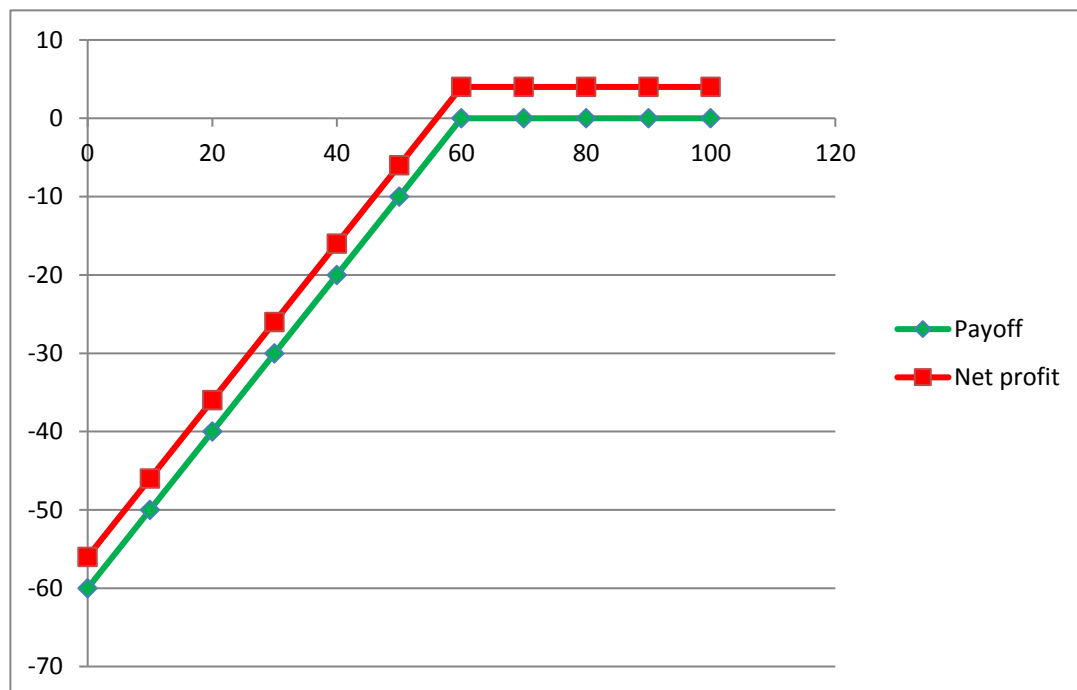
Buy a put

Premium	4										
Strike	60										
S	0	10	20	30	40	50	60	70	80	90	100
Payoff	60	50	40	30	20	10	0	0	0	0	0
Net profit	56	46	36	26	16	6	-4	-4	-4	-4	-4



Sell put

Premium	4										
Strike	60										
S	0	10	20	30	40	50	60	70	80	90	100
Payoff	-60	-50	-40	-30	-20	-10	0	0	0	0	0
Net profit	-56	-46	-36	-26	-16	-6	4	4	4	4	4



Options: Application 1 (In class)

- You buy a call with a strike $K = 150$ and a premium of 10. Calculate the payoff and the net profit of the strategy at maturity.
- You buy a put with a strike $K = 75$ and a premium of 7. Calculate the payoff and the net profit of the strategy at maturity.

8.4 Determinants of Option Value

The price of an option, call or put, depends on a number of factors.

1. **Volatility** in the price of the underlying security: $\uparrow$ volatility $\Rightarrow$ $\uparrow$ probability that the price of the underlying security be greater (call option) or lower (put option) than the strike.

An option can also be seen as an insurance contract against unexpected events. The value of an insurance increases with the volatility in the underlying security.

2. **Maturity**: $\uparrow$ maturity $\Rightarrow$ $\uparrow$ probability that the price of the underlying security reaches a certain level. Time has value.

3. Actual **price** of the underlying security:

- Call option: For a given strike, $\uparrow$ price of security $\Rightarrow$ $\uparrow$ probability that it will be exercised at maturity.
- Put option: For a given strike, $\uparrow$ price of security $\Rightarrow$ $\downarrow$ probability that it will be exercised at maturity.

4. **Strike**

- Call option: For a given price of the underlying security, $\uparrow$ strike $\Rightarrow$ $\downarrow$ probability that it will be exercised at maturity.
- Put option: For given price of the underlying security, $\uparrow$ strike $\Rightarrow$ $\uparrow$ probability that it will be exercised at maturity.

5. **Interest rate**:

- Call option: Purchase (outflow) at a future date of a security at a strike. $\uparrow$ interest rate $\Rightarrow$ $\downarrow$ PV(strike) that the investor will pay.
- Put option: Sale (inflow) at a future date of a security at a strike. $\uparrow$ interest rate $\Rightarrow$ $\downarrow$ PV(strike) that the investor will receive.

The effects of these determinants on the premium of an option is summarized in the following table:

Determinants	Call	Put
Volatility	+	+
Maturity	+	+
Price of security	+	-
Strike	-	+
Interest rate	+	-

Example: Premium of different options on Apple on July 6, 2026. The price of the stock on that day is \$309.67

Contrats	Strike	Maturity 10/07/2026	Maturity 14/08/2026	Maturity 20/11/2026
Call	290	19.30	23.61	34.70
Call	300	8.86	17.50	28.10
Call	310	3.55	11.40	22.71
Call	320	0.88	7.00	17.84
Put	290	0.37	4.42	11.00
Put	300	1.50	7.93	15.61
Put	310	5.25	11.78	19.65
Put	320	12.10	18.10	25.98

As we have seen, there is a link between the price of the underlying security and the strike of the option. We say that at any time t :

- an option is “**in the money**” if the hypothetical exercise of the option generates a positive payoff
 - Call: $S_t > K$
 - Put: $S_t < K$
- an option is “**at the money**” if the hypothetical exercise of the option generates a zero payoff ($S_t = K$)
- an option is “**out of the money**” if the hypothetical exercise of the option generates a zero payoff.
 - Call: $S_t < K$
 - Put: $S_t > K$

8.5 Intrinsic Value and Time Value

The premium of an option can be divided in 2 components:

1. **Intrinsic** value: Value of the option if it is immediately exercised or discarded. This value is always greater or equal to zero.
 - Call
 - $S_t - K$ if $S_t > K$
 - 0 if $S_t \leq K$
 - Put
 - $K - S_t$ if $S_t < K$
 - 0 if $S_t \geq K$
2. **Time** value: Premium - intrinsic value
 - Call: $C_t - \text{Max}[0, S_t - K]$
 - Put: $P_t - \text{Max}[0, K - S_t]$
3. The time value represents the residual value of the option premium which is not explained by the intrinsic value.
4. At maturity, the time value of an option is equal to zero and its premium is equal to the intrinsic value.

Example: Apple

- Date: July 6, 2026
- $S_t = 309.67$
- Call Contract I:
 - Maturity: July 10, 2026
 - Strike: $K = 295$
 - Premium: $C_t = 17.18$
 - Intrinsic value: $\text{Max}[0, S_t - K] = \text{Max}[0, 309.67 - 295] = 14.67$
 - Time value: $C_t - \text{Max}[0, S_t - K] = 17.18 - 14.67 = 2.51$
- Call Contract II:
 - Maturity: September 18, 2026
 - Strike: $K = 295$
 - Premium: $C_t = 25.03$
 - Intrinsic value: $\text{Max}[0, S_t - K] = 14.67$
 - Time value: $C_t - \text{Max}[0, S_t - K] = 25.03 - 14.67 = 10.36$

Remark: One should note that the intrinsic value of the call option is the same in both examples. It is the time value of the option that changes when you change the maturity.

8.6 Option Pricing: The Binomial Model

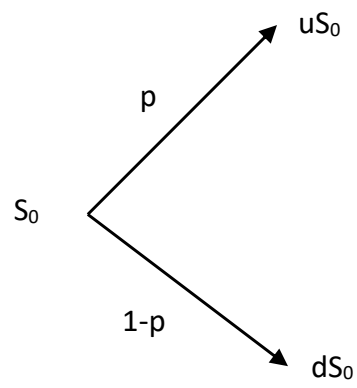
- Let S_0 be the current price of a stock that does not pay dividends
- Next period, S can take only two different values:
 - uS_0 with probability p (UP state)
 - dS_0 with probability $(1 - p)$ (DOWN state)
 - $u = (1 + U)$ where U is the variation rate in the stock price in the high state
 - $d = (1 + D)$ where D is the variation rate in the stock price in the low state
- Let K be the strike
- Let C be the price (premium) of the call option
- Next period, C can take only two different values depending on the evolution of the price of the underlying stock:
 - C_u with probability p (UP state)
 - C_d with probability $(1 - p)$ (DOWN state)
- Agents can borrow or lend at the risk free interest rate, r_f
- $d < (1 + r_f) < u$ (arbitrage restriction)⁴

The evolution of the stock price and the call is depicted in the following graph.

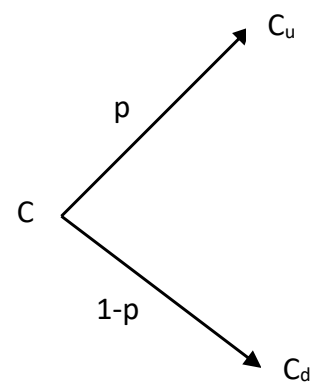
⁴Suppose $d > (1 + r_f)$. At $t=0$, I can borrow at the risk free rate and buy the stock. At $t=1$, the net return of the investment is either $u - (1 + r_f) > 0$ with probability p or $d - (1 + r_f) > 0$ with probability $(1 - p)$. This is an arbitrage opportunity. Suppose $u < (1 + r_f)$. At $t=0$, I can short sell the stock and invest the proceeds in the risk-free asset. At $t=1$, the net return of the investment is either $(1 + r_f) - u > 0$ with probability p or $(1 + r_f) - d > 0$ with probability $(1 - p)$. This is an arbitrage opportunity.

Binomial Tree : 1 period

Stock



Call



Thus, we can write that:

$$C_u = \text{Max } [0, uS_0 - K] \quad \text{prob} = p$$

and

$$C_d = \text{Max } [0, dS_0 - K] \quad \text{prob} = (1 - p)$$

Question: What is the value of the call option today, C ?

Answer: Using the *Law of One Price* (absence of arbitrage), we can replicate the payoff of the call option by creating a portfolio which contains :

- shares of the underlying stock (Δ)
- risk free asset (B) where the investor can lend or borrow at the risk free rate

The idea behind this approach is relatively simple.

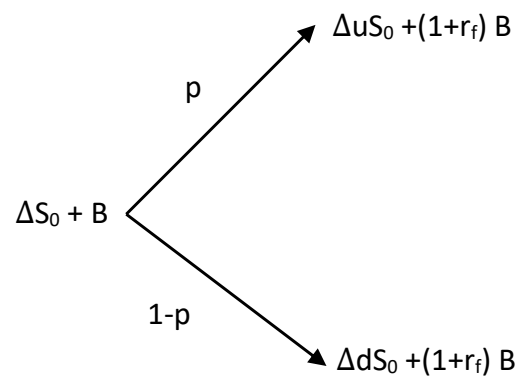
- Although we don't have the price of the call option today, we have its value at maturity in both states (U and D).
- We also have the market price of the stock and the risk-free bond today and in both states at maturity.
- We can then use these two assets (stock and risk-free bond) to create a portfolio that will **replicate** the payoff of the call in each state of nature at maturity.
- In the absence of arbitrage, two assets with the same risk and generating the same payoff must have the same price.
- Thus, the market price of this portfolio of stocks and risk-free bonds must be equal to the price of the call.

Let

- Δ : number of shares of the underlying stock
- B : position in the risk free bond [$B > 0$ (lend) ou $B < 0$ (borrow)]
- ΔuS_0 : value of the stocks in the high state after 1 period
- ΔdS_0 : value of the stocks in the low state after 1 period
- r_f : risk-free interest rate
- $(1 + r_f)B$: value of the investment in the risk-free bond after 1 period

Through replication, the value of the portfolio after one period is equal to the value of the call in each state of nature. Thus, we can write

Replicating portfolio : 1 period



$$\begin{array}{ccc}
\Delta u S_0 + (1 + r_f) B & = & C_u \\
\uparrow & & \uparrow \\
\text{Value replicating} & & \text{Value Call} \\
\text{portfolio (state U)} & & \text{(state U)}
\end{array}$$

and

$$\begin{array}{ccc}
\Delta d S_0 + (1 + r_f) B & = & C_d \\
\uparrow & & \uparrow \\
\text{Value replicating} & & \text{Value Call} \\
\text{portfolio (state D)} & & \text{(state D)}
\end{array}$$

Solving these two equations for the two unknown variables Δ and B , we get,

$$\Delta^* = \frac{C_u - C_d}{(u - d)S_0}$$

and

$$B^* = \frac{C_u - \Delta^* u S_0}{(1 + r_f)}$$

- Δ^* represents the sensitivity of the option's value to the changes in the stock price.
- It is commonly called the **hedge ratio** of the call.
- It measures the number of shares to hold in a portfolio in order to hedge the risk of the underlying asset.

(Δ^*, B^*) is called the **replicating portfolio**. In the absence of arbitrage, the value of the call option today is equal to the cost of the replicating portfolio:

$$C = \Delta^* S_0 + B^*$$

We can develop this method one more step by replacing the value of Δ^* and B^* in the call price equation. We get:

$$\begin{aligned}
C &= \frac{C_u - C_d}{(u - d)} + \frac{C_u - \left[\frac{C_u - C_d}{(u - d)} \right] u}{(1 + r_f)} \\
&= \frac{C_u - C_d}{(u - d)} + \frac{C_u}{(1 + r_f)} - \left[\frac{C_u - C_d}{(u - d)(1 + r_f)} \right] u \\
&= \frac{1}{(1 + r_f)} \left[\frac{C_u(1 + r_f) - C_d(1 + r_f)}{(u - d)} + C_u - \frac{(C_u - C_d)u}{(u - d)} \right] \\
&= \frac{1}{(1 + r_f)} \left[\frac{C_u + r_f C_u - d C_u}{(u - d)} + \frac{u C_d - C_d - r_f C_d}{(u - d)} \right] \\
C &= \frac{1}{(1 + r_f)} \left[C_u \frac{[(1 + r_f) - d]}{(u - d)} + C_d \frac{[u - (1 + r_f)]}{(u - d)} \right]
\end{aligned}$$

Let's define:

$$q = \frac{[(1 + r_f) - d]}{(u - d)} \quad \text{and} \quad (1 - q) = \frac{[u - (1 + r_f)]}{(u - d)}$$

Note: Given the condition that $d < (1 + r_f) < u$, this implies that $0 < q < 1$ and can then be interpreted as a probability, commonly called “risk-neutral” probability.

Thus, we can write:

$$C = \frac{q C_u + (1 - q) C_d}{(1 + r_f)}$$

Given that the call option can take two values after one period, C_u , with probability q , and C_d , with probability $(1 - q)$, the price of a call is equal to the expected future payoffs generated by the call, discounted at the risk free interest rate.

We can write that the expected payoff of a call is equal to:

$$E^q[C] = q C_u + (1 - q) C_d$$

Thus:

$$C = \frac{E^q[C]}{(1 + r_f)}$$

Thus, the value of a call is equal to its expected future payoff discounted at the risk free interest rate, calculated using the “risk-neutral” probability.

Note: One can see that the pricing of the call does not depend on the actual probabilities $[p, (1-p)]$ but on the “risk-neutral” probabilities $[q, (1-q)]$.

Remark: The same analysis prevails for a **put** option. Based on the replicating portfolio approach, we get that

$$\Delta^* = \frac{P_u - P_d}{(u - d)S_0}$$

$$B^* = \frac{P_u - \Delta^* u S_0}{(1 + r_f)}$$

$$P = \Delta^* S_0 + B^*$$

Using the risk-neutral probability approach, we get that:

$$P = \frac{qP_u + (1-q)P_d}{(1 + r_f)}$$

with

$$E^q[P] = qP_u + (1-q)P_d$$

So we can write:

$$P = \frac{E^q[P]}{(1 + r_f)}$$

Example Consider a security with the following characteristics:

- Today: $S_0 = 150$
- In one period: the price of the stock can:
 - increase by 10% (UP state) $\implies uS_0 = 165$
 - decrease by 10% (DOWN state) $\implies dS_0 = 135$
- The probability of the high state (p) is 80%; the probability of the low state ($1 - p$) is 20%.
- $K = 150$
- $r_f = 2\%$ (for the period)

Question: What is the price of **call** option on this stock?

1. Payoff from the call option at maturity in each state of nature.

$$C_u = \text{Max } [0, uS_0 - K] = \text{Max } [0, 165 - 150] = \text{Max } [0, 15] = 15$$

and

$$C_d = \text{Max } [0, dS_0 - K] = \text{Max } [0, 135 - 150] = \text{Max } [0, -15] = 0$$

2. Replicating portfolio (Δ^*, B^*) :

$$\Delta^* = \frac{C_u - C_d}{(u - d)S_0} = \frac{15 - 0}{165 - 135} = 0.5$$

and

$$B^* = \frac{C_u - \Delta^* u S_0}{(1 + r_f)} = \frac{15 - (0.5 \times 165)}{1.02} = -66.18$$

The buyer of the call option can then replicate the payoff of the option by:

1. buying 0.5 of the underlying security
2. borrowing 66.18

Note: We can verify the value of C_u and C_d on the basis of replication:

$$\Delta^* u S_0 + (1 + r_f) B = 0.5(165) - 66.18(1.02) = 15 = C_u$$

and

$$\Delta^* d S_0 + (1 + r_f) B = 0.5(135) - 66.18(1.02) = 0 = C_d$$

Using this result, we can calculate the value of the call today:

$$C = \Delta^* S_0 + B^* = (0.5 \times 150) - 66.18 = 8.82$$

Conclusion: The market price of the call today is 8.82.

An equivalent approach would to calculate the value of the call as the present value of the expected payoff from the call using the “risk-neutral” probabilities.

$$C = \frac{E^q[C]}{(1 + r_f)}$$

with

$$E^q[C] = qC_u + (1 - q)C_d$$

and $C_u = 15$ and $C_d = 0$.

1. Risk neutral probabilities

$$q = \frac{[(1 + r_f) - d]}{(u - d)} = \frac{(1.02 - 0.9)}{(1.1 - 0.9)} = 0.6$$

and

$$(1 - q) = \frac{[u - (1 + r_f)]}{(u - d)} = \frac{(1.1 - 1.02)}{(1.1 - 0.9)} = 0.4$$

2. Expected cash flows from the call option

$$E^q[C] = 0.6(15) + 0.4(0) = 9$$

3. Call premium

$$C = \frac{9}{1.02} = 8.82$$

Remark: One should notice that the price of the call is independent of the probability of the high ($p = 0.8$) and the low ($(1 - p) = 0.2$) state. In addition, one can be tempted to calculate the price of the option using these probabilities, in which case, the expected payoff from the call option would:

$$E^p[C] = 0.8(15) + 0.2(0) = 12$$

and

$$C = \frac{12}{1.02} = 11.76$$

This approach would lead to overestimating the price of the call option.

Imagine a **put** on the same stock with $K = 150$. What would be its price?

1. Payoff from the put option at maturity in each state of nature.

$$P_u = \text{Max } [0, K - uS_0] = \text{Max } [0, 150 - 165] = 0$$

and

$$P_d = \text{Max } [0, K - dS_0] = \text{Max } [0, 150 - 135] = 15$$

2. Replicating portfolio (Δ^*, B^*) :

$$\Delta^* = \frac{P_u - P_d}{(u - d)S_0} = \frac{0 - 15}{165 - 135} = -0.5$$

and

$$B^* = \frac{P_u - \Delta^* u S_0}{(1 + r_f)} = \frac{0 - (-0.5 \times 165)}{1.02} = 80.88$$

The buyer of the put option can then replicate the payoff of the option by:

1. short selling 0.5 of the underlying security
2. lending 80.88

Note: We can verify the value of P_u and P_d on the basis of replication:

$$\Delta^* u S_0 + (1 + r_f) B = -0.5(165) + 80.88(1.02) = 0 = P_u$$

and

$$\Delta^* d S_0 + (1 + r_f) B = -0.5(135) + 80.88(1.02) = 15 = P_d$$

Using this result, we can calculate the value of the call today:

$$P = \Delta^* S_0 + B^* = (-0.5 \times 150) + 80.88 = 5.88$$

Conclusion: The market price of the put today is 5.88.

An equivalent approach would be to calculate the value of the call as the present value of the expected payoff from the call using the “risk-neutral” probabilities.

$$P = \frac{E^q[P]}{(1 + r_f)}$$

with

$$E^q[P] = qP_u + (1 - q)P_d$$

with $P_u = 0$ and $P_d = 15$.

1. Risk neutral probabilities

$$q = \frac{[(1 + r_f) - d]}{(u - d)} = \frac{(1.02 - 0.9)}{(1.1 - 0.9)} = 0.6$$

and

$$(1 - q) = \frac{[u - (1 + r_f)]}{(u - d)} = \frac{(1.1 - 1.02)}{(1.1 - 0.9)} = 0.4$$

2. Expected cash flows from the put option

$$E^q[P] = 0.6(0) + 0.4(15) = 6$$

3. Put premium

$$P = \frac{6}{1.02} = 5.88$$

8.7 Strategies with Options

There are numerous strategies with calls and puts that can be adopted according to the expectations of the investors on the evolution of the underlying security. These strategies generate different payoff profiles.⁵

- **Vertical Spreads:** strategy in which you buy one call (put) option with one strike and sell another call (put) option with a different strike (the same maturity).
 - Bull spreads: the option purchased has a lower strike than the option sold
 - Bear spreads: the option purchased has a higher strike than the option sold
- **Butterfly:** strategy which combines three options of the same kind (either call or puts) with different strikes and same maturity.
- **Straddles:** strategy which combines a call and a put with the same strikes (same maturity).
- **Stangles:** strategy which combines a call and a put with different strikes (same maturity).
- **Ratios:** a straddle strategy in which the number of call options differs from the number of put options.

Example: Straddle: Buy a call and a put at the same strike.

- Buy a call
 - $K = 80$
 - Premium = 4
- Buy a put
 - $K = 80$
 - Premium = 3

⁵See Portait, Poncet (2014), Chap. 12.

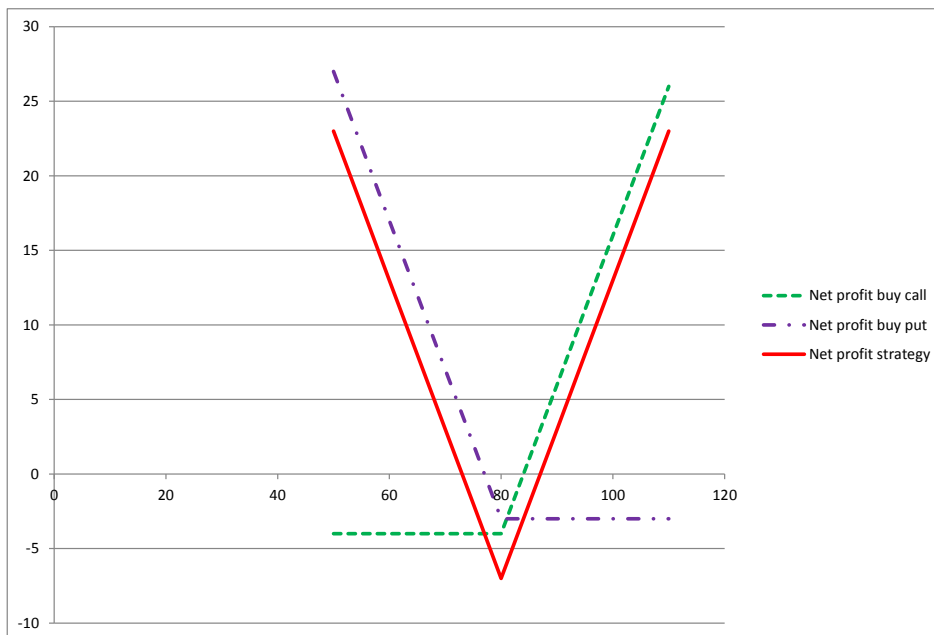
Straddle

1. Buy a call
2. Buy a put

	Buy call	Buy put
Strike	80	80
Premium	4	3

Payoff and net profit

Price	Buy Call		Buy Put		Strategy	
	Payoff	Net profit	Payoff	Net profit	Payoff	Net profit
50	0	-4	30	27	30	23
55	0	-4	25	22	25	18
60	0	-4	20	17	20	13
65	0	-4	15	12	15	8
70	0	-4	10	7	10	3
75	0	-4	5	2	5	-2
80	0	-4	0	-3	0	-7
85	5	1	0	-3	5	-2
90	10	6	0	-3	10	3
95	15	11	0	-3	15	8
100	20	16	0	-3	20	13
105	25	21	0	-3	25	18
110	30	26	0	-3	30	23



Remarks on the straddle.

- When you buy a straddle, you buy two options, a call and a put, but only one option will be exercised. You pay for two contracts in order to exercise only one.
 - If $K > 80$, you exercise the call but not the put.
 - If $K < 80$, you exercise the put but not the call.
- Given that the total cost of the strategy is 7, your net profit is positive when $S > 87$ or $S < 73$ at maturity.
- The straddle pays when the price of the security is either very low or very high.
- This strategy is used when you expect high volatility of the underlying asset.

Options: Application 2 (In class)

Vertical Bull Spread: Buy a call at one strike and sell a call at a higher strike

- **Buy a call**
 - $K = 100$
 - **Premium = 5**
- **Sell a call**
 - $K = 110$
 - **Premium = 2**

Calculate the payoff and the net profit and plot the net profit function for each element and the complete strategy. What expectations on the stock price at maturity would justify using this strategy?

8.8 Beta of options

Consider an option on a stock. What is the relationship between the beta (systematic risk) of the stock and the beta of the option?

Let's come back to the previous example on the pricing of a call.

- $S_0 = 150$
- In one period: the price of the stock can:
 - increase by 10% (UP state) $\implies uS_0 = 165$ ($p = 0.8$)
 - decrease by 10% (DOWN state) $\implies dS_0 = 135$ ($p = 0.2$)
- $K = 150$
- $r_f = 2\%$

- $E(r_m) = 10\%$
- $C = 8.82$

The expected return of the stock is equal to:

$$E(r_s) = 0.8(0.10) + 0.2(-0.1) = 0.06$$

Using CAPM, we can calculate its beta in equilibrium:

$$\begin{aligned} E(r_s) &= r_f + \beta_S[E(r_m) - r_f] \\ 0.06 &= 0.02 + 0.08\beta_S \\ \beta_s &= 0.5 \end{aligned}$$

Consider the call option on that stock with $C_u = 15$, $C_d = 0$ and $C = 8.82$

The expected return of the call is equal to:

$$E(r_c) = 0.8 \left(\frac{15 - 8.82}{8.22} \right) + 0.2 \left(\frac{0 - 8.82}{8.82} \right) = 0.36$$

Using CAPM, we can calculate its beta in equilibrium:

$$\begin{aligned} E(r_c) &= r_f + \beta_c[E(r_m) - r_f] \\ 0.36 &= 0.02 + 0.08\beta_c \\ \beta_c &= 4.25 \end{aligned}$$

$$\implies \beta_c = 8.5 \beta_s$$

- The beta of the call is significantly higher than the beta of the underlying stock.
- The call has more systematic risk than the stock.

Consider the put on the same stock with $P_u = 0$, $P_d = 15$ and $P = 5.88$.

The expected return of the call is equal to:

$$E(r_p) = 0.8 \left(\frac{0 - 5.88}{5.88} \right) + 0.2 \left(\frac{15 - 5.88}{5.88} \right) = -0.49$$

Using CAPM, we can calculate its beta in equilibrium:

$$\begin{aligned} E(r_p) &= r_f + \beta_p [E(r_m) - r_f] \\ -0.49 &= 0.02 + 0.08\beta_p \\ \beta_p &= -6.375 \end{aligned}$$

$$\implies \beta_p = 12.75 \beta_s$$

- The beta of the put is negative since it pays in the DOWN state, that is when the stock price goes down. This effect is reinforced by the fact that there is a very strong probability ($p=0.8$) that the market will go up.
- The put is negatively correlated with the market.

8.9 Leverage with Options

As we have seen, the use of options is not without risk. Suppose that you want to invest in a stock. This can be done in two different ways:

- buying the stocks
- buying options on the stock (leverage)

To illustrate this, let's compare the expected return and the risk (variance) of two strategies.

- buy stocks
- buy call options on the stock

Stock price	90
Investment	90 000
Premium call / share	4
# shares / call	100
Total cost for the call	400
Strike	100

1) Purchase stocks

Stock price at maturity	75	80	85	90	95	100	105	110	115	120	125
# shares	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1 000
Cost	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000
Portfolio value at maturity	75 000	80 000	85 000	90 000	95 000	100 000	105 000	110 000	115 000	120 000	125 000
Return	-17%	-11%	-6%	0%	6%	11%	17%	22%	28%	33%	39%

Exp. return	11%
Variance	3,09%

2) Purchase call options

Stock price at maturity	75	80	85	90	95	100	105	110	115	120	125
# calls	225	225	225	225	225	225	225	225	225	225	225
# shares under control	22 500	22 500	22 500	22 500	22 500	22 500	22 500	22 500	22 500	22 500	22 500
Cost of calls	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000	90 000
Portfolio value at maturity	0	0	0	0	0	0	112 500	225 000	337 500	450 000	562 500
Return	-100%	-100%	-100%	-100%	-100%	-100%	25%	150%	275%	400%	525%

Exp. return	70%
Variance	490,7%

Remarks:

- The portfolio of stocks is composed of 1 000 shares.
- The portfolio of options is composed of 225 calls but since each call gives the right to buy 100 shares, an investor has potentially the control of 22 500 shares.

Conclusions:

- The expected return and the volatility of the portfolio with options are significantly higher than for the portfolio with stocks.
- The return of the portfolio with stocks varies smoothly with the stock price at maturity.
- The return of the portfolio with options exhibits a jump as the stock price varies.
 - If $S_T < 100$, the option is not exercised and you lose all (100%) your initial investment.
 - If $S_T > 100$, the option is exercised and you make profit on each share. Given that you can buy 22 500 shares, the return increases significantly.
 - Option strategies are risky!!!

Note: According to some studies, around 75 - 80% of options are not exercised at maturity.

9 Capital Structure Theory

Firms have different sources of financing:

1. Equity
2. Debt
3. Savings (retained earnings)

Some Figures

FIGURE 15.5

Net External Financing and Capital Expenditures by U.S. Corporations, 1975–2014

In aggregate, firms have raised external capital primarily by issuing debt. These funds have been used to retire equity and fund investment, but the vast majority of capital expenditures are internally funded. (Amounts adjusted for inflation to reflect 2014 dollars.)

Source: Federal Reserve, *Flow of Funds Accounts of the United States*, 2014.

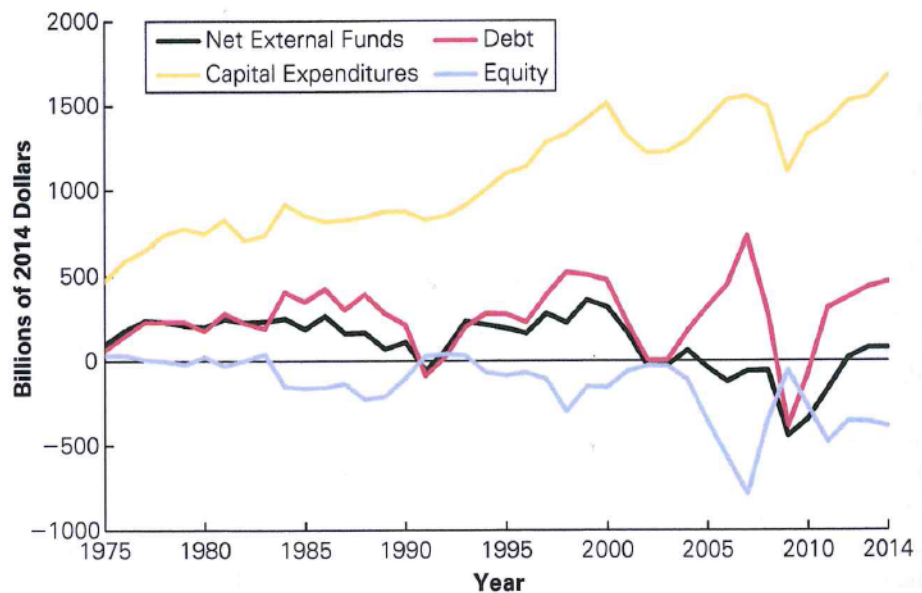


FIGURE 15.6

Debt-to-Value Ratio $[D/(E + D)]$ of U.S. Firms, 1975–2014

Although firms have primarily issued debt rather than equity, the average proportion of debt in their capital structures has not increased due to the growth in value of existing equity.

Source: Compustat and Federal Reserve, *Flow of Funds Accounts of the United States*, 2014.

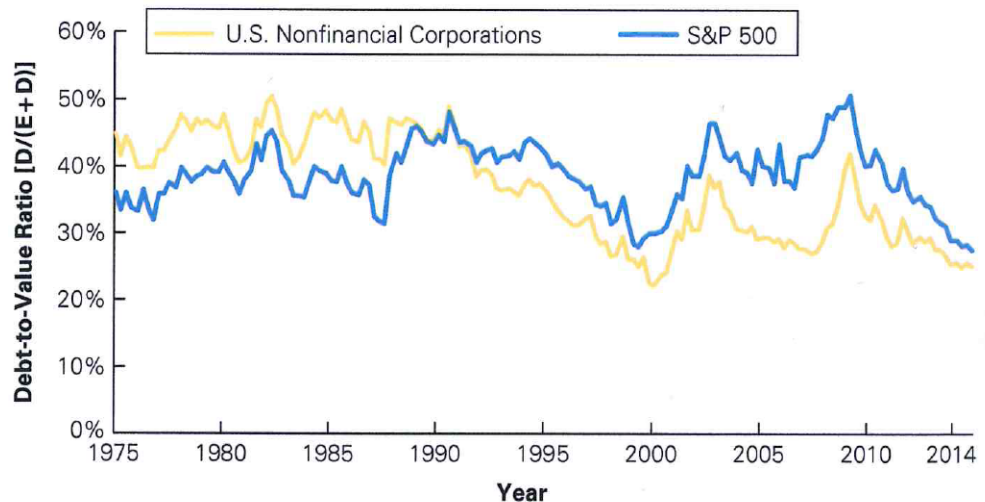
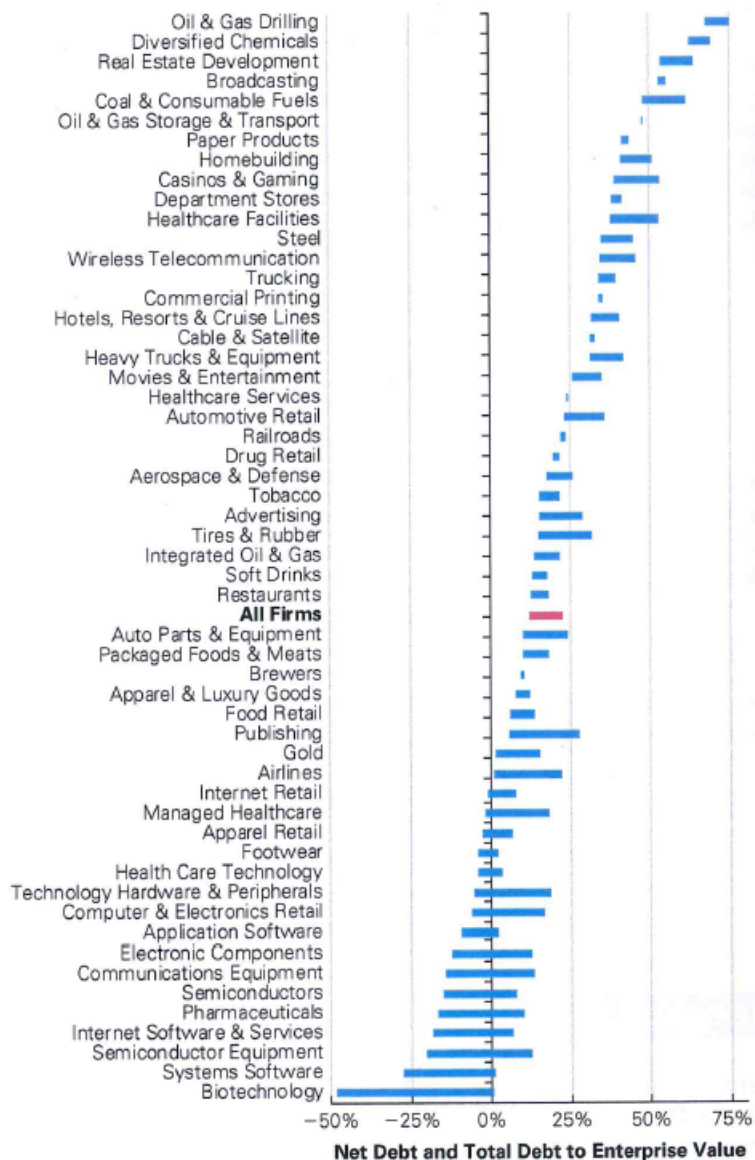


FIGURE 15.7

Debt-to-Enterprise Value Ratio for Select Industries (2015)

Figure shows industry median levels of net debt and total debt as a percentage of firm enterprise value. The spread between them, shown by the blue bars, corresponds to cash holdings. For example, Biotech firms tend to have no debt but hold a great deal of cash, and so have negative net debt. Oil & Gas Drilling firms have much less cash and over 75% total debt. While the median level of debt for all U.S. stocks was about 23% of firm value, note the large differences by industry.

Source: Capital IQ, 2015.



Assumption: Managers maximize the value of the firm for equityholders.

Question: Does the choice of a firm's capital structure (equity vs. debt) has an impact on its value?

$\Rightarrow$ Is there an optimal capital structure which maximizes the firm's value?

A first answer to this question was brought by the work of F. Modigliani & M. Miller who showed that under the assumption of complete and perfect capital markets, the value of a firm is **independent** of its capital structure. Any combination of securities will maximize the value of the firm!

Remark: The capital structure analysis is conducted for a fixed investment policy.

9.1 Firm Capital Structure with Perfect Capital Market

Assumptions of Modigliani et Miller (M&M) analysis:

1. no transaction costs
2. no taxation
3. no bankruptcy costs
4. firms and investors can lend and borrow at the same risk free rate
5. all firms face the same operational risk
6. no agency costs (managers are assumed to maximize the value of the firm)
7. no information asymmetry (all agents have the same information)
8. all assets and securities are expressed at their market value
9. all cash flows are perpetuities

Remark: The analysis also applies to finite horizons or if the interest rate varies over time and the debt is risky as long as the debt is fairly priced by the market [See Berk & De Marzo (page 483)]. However, the analytical solutions may differ.

Notation:

- V : firm value
- V_a : market value of assets
- V_e : market value of equity
- V_d : market value of debt
- $E(\tilde{r}_e)$: expected return on equity
- r_f : return on risk free debt
- $E(\tilde{r}_0)$: expected return on equity of an unlevered firm (no debt)

Proposition I (M&M I): If capital markets are perfect, the value of a firm is independent of its capital structure:

$$V^l = V^u$$

where u stands for the unlevered firm and l for the levered firm.

Intuitive proof: Suppose perfect capital markets and two firms that are identical in all aspects except for their capital structure.

- Firm U (unlevered) has no debt and 100% equity: $V^u = V_e^u$
- Firm L (levered) has debt and equity: $V^l = V_d^l + V_e^l$

Question: In which firm (U or L) would an investor prefer to invest?

To answer this question, let's consider alternative investment strategies where the EBIT represents the operational cash flows.

1. **Strategy 1:** Buy 1% of firm U.

Firm U is 100% equity financed so you buy only the equity.

Securities	Investment	Cash flows
Equity	$0.01V_e^u$	0.01 EBIT
Strategy	$0.01V^u$	0.01 EBIT

Note: Equityholders of 100% equity financed firm receive all the firm cash flows (EBIT).

2. **Strategy 2:** Buy 1% of firm L.

Firm L has both debt and equity so you need to buy 1% of the debt and 1% of the equity.

Securities	Investment	Cash flows
Debt	$0.01V_d^l$	0.01 interests
Equity	$0.01V_e^l$	0.01 (EBIT - interests)
Strategy	$0.01(V_d^l + V_e^l)$ $= 0.01V^l$	0.01 EBIT

Notes:

- Given that bonds are perpetuity, they pay interests every year to perpetuity and the capital is never reimbursed.
- Holding 1% of the bonds gives the investor the right to receive 1% of all the interest paid every year.

- Holding 1% of equity gives the investor the right to receive 1% of equity cash flows which are equal to the firm's cash flows after the payment of interests (FCF - interests).

Conclusions:

- The two strategies offer the same cash flows and have the same risk.
- In the absence of arbitrage, the cost of these two strategies must be the same.
- Thus, $V^u = V^l$
- The value of the unlevered firm (without debt) is equal to the value of the levered firm (with debt)

Consider two alternative strategies.

1. **Strategy 3:** Buy 1% of equity of firm L.

Securities	Investment	Cash flows
Equity	$0.01V_e^l$	0.01 (EBIT - interests)
Strategy	$0.01(V^l - V_d^l)$	0.01 (EBIT - interests)

2. **Strategy 4:**

- Borrow an amount equal to 1% of the debt of firm L
- Buy 1% of firm U

Securities	Investment	Cash flows
Debt (Borrow)	$-0.01V_d^l$	- 0.01 interests
Equity	$0.01V_u$	0.01 EBIT
Strategy	$0.01(V_u - V_d^l)$	0.01 (EBIT - interests)

Notes:

- The negative sign on the debt strategy is explained by the fact that it is the inverse strategy as buying the debt (lending).
- Given that investors and firms borrow at the same rate, if an investor borrows an amount equal to 1% of the debt of firm L, he will pay back the same amount of interests as the firm.

Conclusions:

- The two strategies offer the same cash flows and have the same risk.
- In the absence of arbitrage, the cost of these two strategies must be the same.
- Thus, $V^u = V^l$

Another way to state Proposition I is the following:

$$V_a = V_e + V_d = V^u$$

- Under the assumption of perfect capital markets, the cash flows to stakeholders (equity and debt) is equal to the total cash flows generated by the firm's assets.
- According to the Law of one Price, as long as the securities issued by the firm does not impact on the firm's cash flows, the total market value of the securities must be equal to the market value of the assets.
- This implies that by holding a portfolio of debt and equity, any investor can replicate the cash flows generated by the equity of an unlevered firm.
- Thus, the value of a firm does not depend on its capital structure but rather on the market value of its assets or the cash flows generated by the assets.

Proposition 1 has two implications.

1. Investment and financing decisions are independent

Example: Consider a firm with an investment project generating a risk free cash flow of \$12 M in a year. The cost of the project is \$10 M. This project can be financed in two different ways: i) self-financing ii) issue of risk free security for \$5 M. The risk free rate of return is 3%.

Question: Should the firm issue the new security to finance the project?

a) Self-financing

$$NPV = -10 + \frac{12}{1.03} = 1.65$$

b) Issue of security

If the firm issues for \$5 M of the new security, the original shareholders must invest only \$5M in the project. However, they must pay back \$5.15 M in a year, leaving a net cash flow of \$6.85 M to them. The NPV for the shareholders is then:

$$NPV = -5 + \frac{6.85}{1.03} = 1.65$$

Conclusion: The investment decision is independent of the financing decision. The NPV does not depend on how the project is financed.

2. Weighted average cost of capital (WACC) is independent of a firm's capital structure

$$WACC^l = \frac{V_d}{V} r_f + \frac{V_e}{V} E(\tilde{r}_e) = WACC^u$$

As we have seen earlier, the value of an asset is equal to the present value of the cash flows generated by this asset. Thus, we can write:

$$V = \sum_{t=0}^{\infty} \frac{\text{Firm Cash Flows}}{(1 + \text{Cost of Capital})^t} = \sum_{t=0}^{\infty} \frac{\text{FCF}}{(1 + WACC)^t}$$

Intuition:

- We have seen in the capital budgeting section that the the firm's cash flows do not depend on its capital structure
- Proposition I of M&M shows that the value of the firm does not depend on its capital structure.
- If the firm's value and the firm's cash flows do not depend on its capital structure, then it must be that the WACC is also independent of the firm's capital structure.

Although the cost of capital (WACC) does not depend on the firm's capital structure, the cost of equity does.

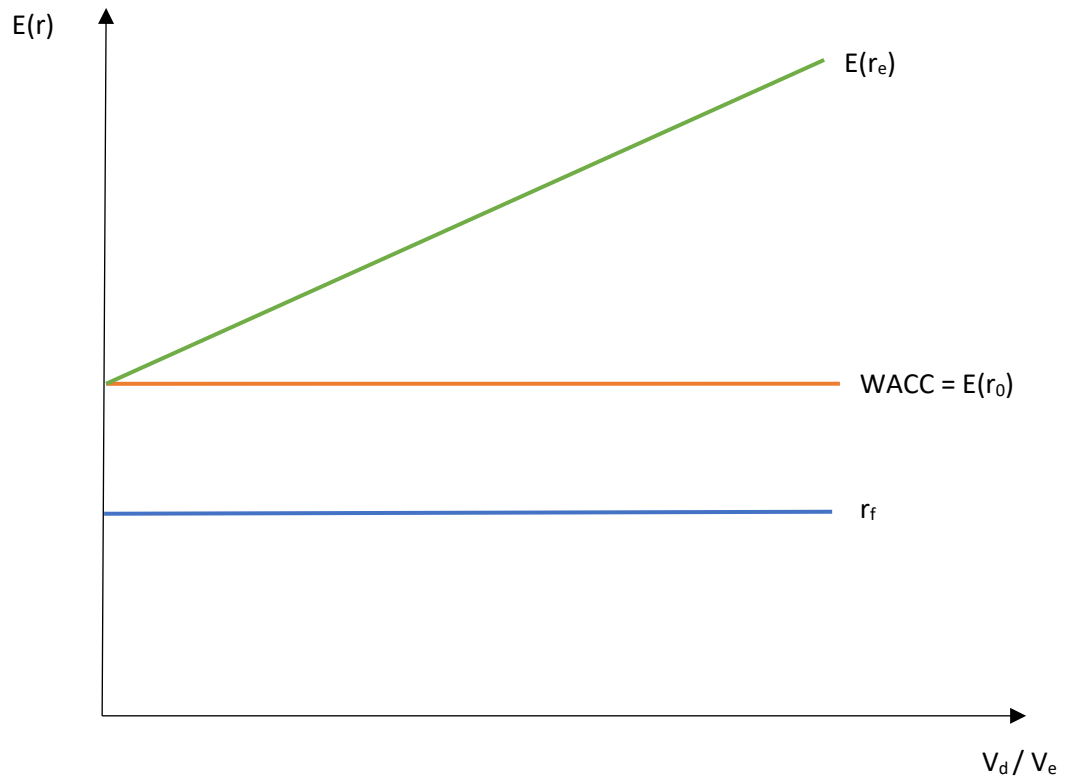
Proposition II (M&M II): The expected return on equity increases proportionally with the ratio of debt to equity (gearing ratio) expressed in market value.

$$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e} [E(\tilde{r}_0) - r_f]$$

with $E(\tilde{r}_0) = E(\tilde{r}_e^u)$ (expected return on equity of an unlevered firm)

Proposition II says that the expected return on equity of a levered firm is equal to the expected return on equity of an unlevered firm plus a premium which is proportional to the market value of debt to equity ratio. This is what we call the *financial leverage effect*. Graphically, we get:

Proposition II (M&M)



Example

	U	L
Assets	8 000	8 000
Debt	0	4 000
Equity	8 000	4 000
Risk-free rate	5%	5%
EBIT / year	1 200	1 200

1. Calculate the expected return on equity for U and L

- According to Proposition II,

$$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e} [E(\tilde{r}_0) - r_f]$$

- Firm U

We first need to calculate $E(\tilde{r}_0)$ which is equal to the cost of equity of the unlevered firm. Knowing that cash flows are perpetuities, we can write for firm U:

$$V_e^u = \frac{EBIT}{E(\tilde{r}_0)} \Rightarrow E(\tilde{r}_0) = \frac{EBIT}{V_e^u}$$

Thus, $E(\tilde{r}_0) = E(\tilde{r}_e^u) = 1\,200 / 8\,000 = 0.15$

- Firm L

$$E(\tilde{r}_e^l) = 0.15 + \frac{4\,000}{4\,000} [0.15 - 0.05] = 0.25$$

2. Calculate the WACC for U and L.

- Firm U

$$WACC^u = \frac{0}{8\,000} (0.05) + \frac{8\,000}{8\,000} (0.15) = 0.15$$

- Firm L

$$WACC^l = \frac{4\,000}{8\,000} (0.05) + \frac{4\,000}{8\,000} (0.25) = 0.15$$

9.1.1 Leverage and Risk: A First Approach

Proposition II states that the expected return on equity increases with leverage. Let's illustrate this effect with the following example.

- Consider a project over 1 year which is 100% equity financed. The cash flows are:

– Investment: 1 000 (t=0)

- Cash flows (High state): 1 400 (t=1)
- Cash flows (Low state): 900 (t=1)
- The probability of each state is 50%
- Realized return on equity at $t = 1$.
 - High state: $r_e^H = (1\,400 - 1\,000)/1\,000 = 40\%$
 - Low state: $r_e^L = (900 - 1\,000)/1\,000 = -10\%$
- Expected return on equity at $t = 0$.

$$E(\tilde{r}_e) = 0.5r_e^L + 0.5r_e^H = 15\%$$

Assume an alternative capital structure to finance this project which is now financed with 500 of risk-free debt at the risk-free rate of 5%. Does this impact on the cost of equity?

- First, let's examine the cash flows associated to this new capital structure.

		High	Low
	t= 0	t=1	t=1
Debt	500	525	525
Equity	500	875	375
Firm	1 000	1 400	900

- Realized return on equity at $t = 1$.
 - High state: $r_e^H = (875 - 500)/500 = 75\%$
 - Low state: $r_e^L = (375 - 500)/500 = -25\%$
- Expected return on equity at $t = 0$.

$$E(\tilde{r}_e) = 0.5r_e^L + 0.5r_e^H = 25\%$$

Conclusion: $\uparrow$ leverage $\Rightarrow \uparrow E(\tilde{r}_e)$

Question: Why?

Explanation. Let's look at the variation in returns between the two states.

	Δ return (H - L)	Risk premium ($E(\tilde{r}) - r_f$)
Debt	5% - 5% = 0%	5% - 5% = 0%
Unlevered equity	40% - (-10)% = 50%	15% - 5% = 10%
Levered equity	75% - (-25)% = 100%	25% - 5% = 20%

Conclusions:

- The return on equity is more sensitive to changes in the state of the economy when the firm uses leverage.
- This type of risk is simply **systematic risk**.
- Equity holders will ask for a higher expected return, thus a risk premium, to compensate for the increased risk associated to leverage.

An alternative way to analyze the link between the leverage effect and systematic risk is to use the CAPM.

- Suppose an asset financed by debt and equity.
- The beta of this asset is equal to the weighted average of the beta of each security

$$\beta_a = \frac{V_d}{V} \beta_d + \frac{V_e}{V} \beta_e$$

Given that $V = V_d + V_e$, we can solve for β_e

$$\beta_e = \beta_a + \frac{V_d}{V_e} [\beta_a - \beta_d]$$

- One can see a clear parallel between this relationship and Proposition II on the expected return on equity $[E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e} [E(\tilde{r}_0) - r_f]]$
- An increase in leverage increases the beta of equity and thus, by CAPM, increases the expected return on equity.

9.1.2 Firm Value and Arbitrage

Question: What happens if $V^l \neq V^u$?

Answer: According to M&M, there will be an arbitrage opportunity that investors will exploit to make a profit. This will continue until the two firms have the same value.

General setting: Suppose that $V^l > V^u$. We can elaborate the following arbitrage strategy:

- Short sell the equity of L
- Borrow an amount equal to the debt of L
- Buy the equity of U

The cash flows are as follows:

	$t = 0$	$t > 0$
Short sell equity L	V_e^l	$-[EBIT - r_f V_d^l]$
Borrow	V_d^l	$-r_f V_d^l$
Buy equity U	$-V^u$	EBIT
Cash flows	$(V_e^l + V_d^l) - V^u > 0$ $= V^l - V^u > 0$	0

- We started from the assumption that $V^l > V^u$.
- There exists an arbitrage strategy which allows an investor to make a positive profit without taking any risk.
- Thus, investors will short sell the stocks of L, which will drive down its price, and buy the stocks of U, which will drive up its price.
- This will continue until we get back to the equilibrium situation where $V^l = V^u$ and that the arbitrage opportunity has disappeared.

Example: Consider two firms with the same cash flows.

- $E(\text{EBIT}) = 500\,000$
- Same operational risk
- Only difference is their capital structure
- Value of debt of L: 1 000 000
- Agents can borrow or lend at the same risk free-rate: 3%.
- Interests on debt paid by L: $r_f \times V_d = 30\,000$
- $E(r_e^u) = 0.12$
- $E(r_e^l) = 0.14$

Question 1: Is Proposition I of M&M validated?

Answer 1: Based on Proposition 1: $V^l = V^u$. Let's calculate the value of both companies.

$$V^u = \frac{E(\text{EBIT})}{E(r_e^u)} = \frac{500\,000}{0.12} = 4\,166\,667$$

$$V^l = V_d^l + V_e^l = 1\,000\,000 + \frac{E(\text{EBIT}) - r_f V_d}{E(r_e^l)} = 1\,000\,000 + \frac{470\,000}{0.14} = 4\,357\,143$$

Firm L is overvalued relative to firm U . Proposition 1 is not validated.

Question 2: Is there an arbitrage opportunity for investors?

Answer 2: Given that $V^l > V^u$, we can short sell the equity of L , buy the equity of U and use home leverage in order to generate a positive profit. Consider the following strategy:

- Borrow an amount equal to 1% of the debt of L at the risk-free rate of 3%
- Short sell 1% of equity of L
- Buy 1% of equity of U

The cash flow structure will be the following:

Strategy	$t = 0$	$t > 0$
Borrow	$0.01V_d^l$	$-0.01r_f V_d^l$
Short sell 1% equity of L	$0.01V_e^l$	$-0.01[EBIT - r_f V_d^l]$
Buy 1% equity of U	$-0.01V_e^u$	$0.01EBIT$
Cash flows	$0.01[(V_e^l + V_d^l) - V^u]$	0

Strategy	$t = 0$	$t > 0$
Borrow	10 000	$-10\,000(0.03) = -300$
Short sell 1% equity of L	33 571	$-470\,000(0.01) = -4\,700$
Buy 1% equity of U	-41 667	$500\,000(0.01) = 5\,000$
Total Cash Flows	1 904	0

Note 1: The value of equity of L is equal to: $V^l - V_d^l = 3\,357\,143$.

Note 2: The same example can be done with any percentage of debt and equity. There exists many arbitrage strategies. The important point is to respect two conditions: i) invest no wealth and ii) make a profit with certainty.

The outcome of this process is that the stock price of L will go down while the stock price of U will go up until the stock price of both firms are equal.

9.2 Market Imperfection 1: M&M with Taxes

An important assumption in the previous section is the absence of corporate taxes. Yet, we know that in reality, corporations are taxed on their taxable income.

Suppose that corporations are taxed: $\tau > 0$. How does that impact on the firm's capital structure decision?

- When a firm issues debt, it pays interests to creditors.
- These interests are tax deductible for the firm and thus, provide an advantage compared to equity financing.
- In the presence of taxes, the value of a levered firm is then equal to its value without leverage plus the tax shield associated to debt financing. In other words,

$$\begin{array}{ccccc}
 V^l & = & V^u & + & \frac{\tau r_f V_d}{r_f} \\
 \uparrow & & \uparrow & & \uparrow \\
 \text{Value levered} & & \text{Value unlevered} & & \text{Tax shield} \\
 \text{firm} & & \text{firm} & &
 \end{array}$$

Remarks:

- The **tax shield** is equal to the present value of the tax savings on interests ($\tau r_f V_d$) discounted at the risk-free rate since debt is risk-free.
- This formula is valid only if the cash flows are perpetuities.

We can then rewrite Proposition I as follows:

Proposition I (with taxes): The value of a firm increases with the market value of debt:

$$V^l = V^u + \tau V_d$$

Conclusion: In the presence of corporate taxes, the value of a firm increases linearly with leverage.

Implication of Proposition I (with taxes): The **weighted average cost of capital** (WACC) decreases with the proportion of debt in the firm's capital structure:

$$\text{WACC} = \frac{V_d}{V} r_f (1 - \tau) + \frac{V_e}{V} E(\tilde{r}_e)$$

Another way to see this result is the following.

- As previously mentioned, the the firm's cash flows do not depend on its capital structure
- Proposition I of M&M (with taxes) shows that the value of the firm increases with the amount of debt in its capital structure.
- If the firm's cash flows do not depend on its capital structure but the firm's value increases with leverage, then it must be that the WACC is decreasing with leverage.

Proposition II (with taxes):

$$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e} [E(\tilde{r}_0) - r_f](1 - \tau)$$

There exists an alternative version of the WACC in the M&M setting. Replacing expected return as defined in Proposition II (with taxes) in the WACC, we get:

$$\text{WACC} = E(\tilde{r}_0) \left[1 - \tau \frac{V_d}{V} \right]$$

Hence if $\tau = 0 \Rightarrow \text{WACC} = E(\tilde{r}_0) = E(\tilde{r}_e^u)$.

Remark: It is assumed that the debt is risk-free. This does not have to hold and it can be shown that Proposition 1 (without and with taxes) hold if corporate debt is risky as long as there is no bankruptcy costs (See Copeland & Weston, Chapter 13).

Capital Structure: Application 1 (In class)

- A firm reports an annual EBIT of 153.85 (M) per year at perpetuity and the corporate tax rate is 35%.
 - Assume that the firm's net cash flows are paid in dividends every year.
 - $E(\tilde{r}_e^u) = 0.10$
1. Suppose that $V_d = 0$. Calculate the value of the firm and equity as well as the cost of equity and the WACC.
 2. Suppose that the firm changes its capital structure and issues risk-free debt to buy back equity: $V_d = 400$ and $r_f = 0.04$. Determine the impact on the value of the firm and equity as well as the cost of equity and the WACC.

9.2.1 Market Imperfection II: M&M with Taxes and Bankruptcy Costs

A second important assumption in the previous analysis is the absence of bankruptcy costs. Firms can go bankrupt but this procedure is costless. We know this not true in reality and that an increase in leverage raises the probability of bankruptcy and hence, the cost of bankruptcy.

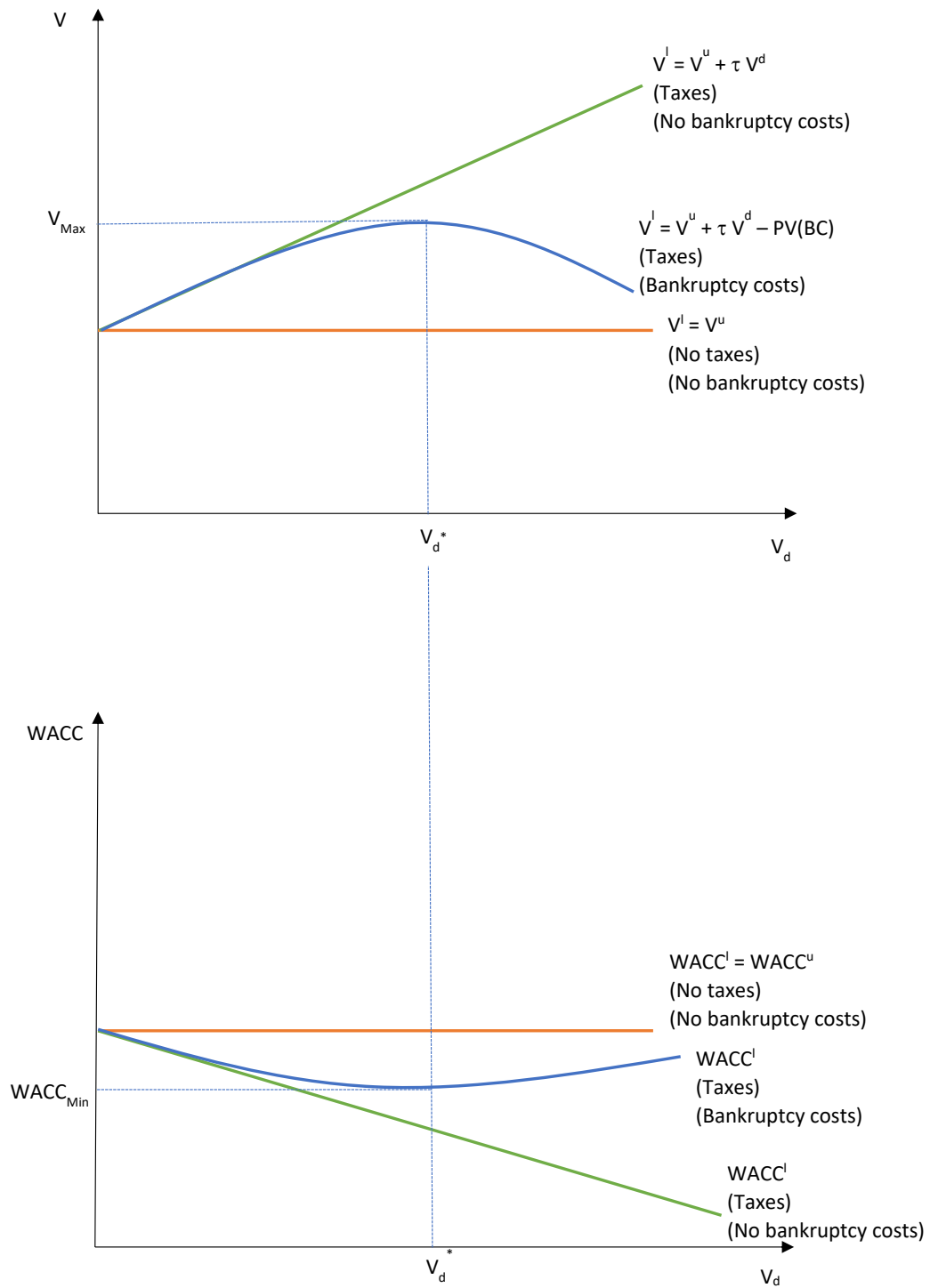
In the presence of taxes and bankruptcy costs, we can show that there exists an optimal capital structure resulting from the compromise between the benefit (tax deductibility of interests) and the cost (bankruptcy costs) of debt financing. Thus, we can write:

$$V^l = V^u + \tau V_d - \text{PV}(\text{bankruptcy costs})$$

Remarks:

1. This approach to valuation is commonly referred to the **Adjusted Present Value (APV)** or the **Trade-Off Theory**.
2. The capital structure which maximizes the value of the firm is the same as the one which minimizes the WACC.
3. Indeed, for given cash flows, minimizing the WACC is equivalent to maximizing the value of the firm.
4. See Graph below.

Adjusted Present Value



9.3 Leverage and Systematic Risk

We have seen two equilibrium models to calculate the cost of different securities:

- CAPM
- M&M

Securities	CAPM	M&M
Debt	$E(\tilde{r}_d) = r_f + \beta_d[E(\tilde{r}_m) - r_f]$	$E(\tilde{r}_d) = r_f \quad (\beta_d = 0)$
Unlevered equity	$E(\tilde{r}_e^u) = r_f + \beta_e^u[E(\tilde{r}_m) - r_f]$	$E(\tilde{r}_e^u) = E(\tilde{r}_0)$
Levered equity	$E(\tilde{r}_e^l) = r_f + \beta_e^l[E(\tilde{r}_m) - r_f]$	$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d^l}{V_e^l} [E(\tilde{r}_0) - r_f](1 - \tau)$
Cost of capital	$WACC = \frac{V_d}{V} r_f (1 - \tau) + \frac{V_e}{V} E(\tilde{r}_e)$	$WACC = E(\tilde{r}_0) \left[1 - \tau \frac{V_d}{V} \right]$

9.3.1 Levered and Unlevered Equity Beta

- As we have seen earlier, a change in the firm's capital structure will have an impact on the market risk faced by equityholders.
- We can show that there exists a relationship between the systematic risk of an unlevered firm (β^u) and the systematic risk of the same firm with leverage (β_e^l).

Case 1: Taxes ($\tau > 0$) and debt is risk-free ($\beta_d = 0$ and $E(r_d) = r_f$)

Setting equal the definition of $E(r_e)^l$ for CAPM and M&M and using the fact that:

$$E(\tilde{r}_0) = E(\tilde{r}_e^u) = r_f + \beta_e^u[E(\tilde{r}_m) - r_f]$$

we get:

$$\begin{aligned}
 r_f + \beta_e^l [E(\tilde{r}_m) - r_f] &= E(\tilde{r}_0) + \frac{V_d}{V_e} [E(\tilde{r}_0) - r_f](1 - \tau) \\
 &= r_f + \beta^u [E(\tilde{r}_m) - r_f] + \frac{V_d}{V_e} [r_f + \beta^u [E(\tilde{r}_m) - r_f]](1 - \tau) \\
 \beta_e^l [E(\tilde{r}_m) - r_f] &= [E(\tilde{r}_m) - r_f] \left[1 + (1 - \tau) \frac{V_d}{V_e} \right] \beta^u \\
 \beta_e^l &= \left[1 + (1 - \tau) \frac{V_d}{V_e} \right] \beta^u
 \end{aligned}$$

Case 2: Taxes ($\tau > 0$) and risky debt ($\beta_d > 0$)

$$\beta_e^l = \left[1 + (1 - \tau) \frac{V_d}{V_e} \right] \beta^u - \frac{V_d}{V_e} (1 - \tau) \beta_d$$

Case 3: No taxes ($\tau = 0$) and debt is risk-free ($\beta_d = 0$ and $E(r_d) = r_f$)

Using the result from Case 1 and assuming $\tau = 0$, we get:

$$\beta_e^l = \left[1 + \frac{V_d}{V_e} \right] \beta^u$$

Example:

Consider a firm with the following information:

- $V_d = 100M$
- $V_e = 200M$
- $V_d/V = 0.333$
- $V_e/V = 0.667$
- $V_d/V_e = 0.5$
- $\tau = 25\%$
- $\beta_e^l = 1.2$
- $r_f = 0.03$
- $E(r_m) = 0.10$
- Risk-free debt

1. Calculate the actual cost of equity.

Using CAPM, we can write:

$$E(\tilde{r}_e^l) = r_f + \beta_e^l [E(\tilde{r}_m) - r_f] = 0.03 + 1.2[0.10 - 0.03] = 0.114$$

2. Calculate the WACC.

Again, using CAPM, we can write:

$$\begin{aligned} WACC &= \frac{V_d}{V} r_f (1 - \tau) + \frac{V_e}{V} E(\tilde{r}_e) \\ &= \frac{100}{300} (0.03) (1 - 0.25) + \frac{200}{300} (0.114) \\ &= 0.0835 \end{aligned}$$

3. Assume that the firm decides to modify its capital structure and increase its ratio to $V_d/V_e = 0.75$. What is the impact on the cost of equity?

- We have seen that there exist a relationship between the beta unlevered and levered equity. Yet, we don't have a relationship between the levered betas with two different levels of leverage.
- The solution is to use the unlevered beta as a transition point from one levered beta to another.
- First, we calculate the unlevered beta of the firm with the existing capital structure.

$$\beta^u = \frac{\beta_e^l}{\left[1 + (1 - \tau) \frac{V_d}{V_e}\right]} = \frac{1.2}{[1 + (1 - 0.25)(0.5)]} = 0.873$$

- Second, we calculate the new levered beta with the new capital structure:

$$\beta_e^l = \left[1 + (1 - \tau) \frac{V_d}{V_e}\right] \beta^u = [1 + (1 - 0.25)(0.75)] 0.873 = 1.36$$

- Third, we calculate the new cost of equity using CAPM.

$$E(\tilde{r}_e^l) = r_f + \beta_e^l [E(\tilde{r}_m) - r_f] = 0.03 + 1.36[0.10 - 0.03] = 0.1252$$

9.4 Gross Debt vs. Net Debt

- The Modigliani & Miller analysis relies on the fact that securities (equity and debt) are valued at their fair market price.
- For simplicity, we often assume that firm's have target ratios of debt and equity.
- However, that are many situation where we need to calculate the market value of equity and debt.
- This raises a concern on how we value debt, that is whether we consider the debt to be net or gross of cash or cash equivalent securities.
- Indeed, as we know, the assets on a firm's balance sheet include cash or cash-like securities and they reduce the risk of the firm.

Question: How do we account for cash in M&M valuation analysis?

Answer:

- Given that cash and cash-like securities reduce risk, they should be taken into account in the analysis.
- This will be done by calculating the value of **net** debt, that is the market value of debt (gross) minus the value of cash and cash-like securities.

10 Firm Valuation and Financial Leverage

We have seen that the value of a firm is equal to the present value of the firm's cash flow discounted at the weighted average cost of capital (WACC). Although the firm's cash flows are independent of its capital structure, the WACC is affected by a variation in the firm's financial structure. Hence, for a given sequence of cash flows, the objective of maximizing the value of the firm can be achieved by minimizing the WACC.

Let's consider three different scenarios:

- Debt is risk free ($r = r_f$) and no taxes.
- Debt is risk free ($r = r_f$) and taxes.
- Debt is risky (cost of debt financing increases with the proportion of debt in the firm's capital structure) and taxes.

Before analyzing these scenarios, let's consider the cost of equity financing.

10.1 Cost of Equity

The calculation of the cost of equity as a function of financial leverage can be conducted using two different but equivalent methods:

Method 1: CAPM

Step 1: Obtain the information on actual beta of equity, β_e^l , and the V_d/V_e ratio.

Step 2: Estimate the unlevered beta of equity under the assumption that the firm had no debt: β^u :

$$\beta^u = \frac{\beta_e^l}{\left[1 + \frac{V_d}{V_e}\right]} \quad (\text{Without taxes})$$

$$\beta^u = \frac{\beta_e^l}{\left[1 + (1 - \tau)\frac{V_d}{V_e}\right]} \quad (\text{With taxes})$$

Step 3: Using β^u , estimate the levered beta of equity, β_e^l , for each V_d/V_e ratio:

$$\beta_e^l = \beta^u \left[1 + \frac{V_d}{V_e}\right] \quad (\text{Without taxes})$$

$$\beta_e^l = \beta^u \left[1 + (1 - \tau)\frac{V_d}{V_e}\right] \quad (\text{With taxes})$$

Step 4: Estimate the cost of equity for each V_d/V_e ratio:

$$E(\tilde{r}_e^u) = r_f + \beta^u [E(\tilde{r}_m) - r_f]$$

$$E(\tilde{r}_e^l) = r_f + \beta_e^l [E(\tilde{r}_m) - r_f]$$

Method 2: Modigliani & Miller:

$$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e} [E(\tilde{r}_0) - r] \quad (\text{Proposition II without taxation})$$

$$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e} [E(\tilde{r}_0) - r](1 - \tau) \quad (\text{Proposition II with taxation})$$

with $E(\tilde{r}_0) = E(\tilde{r}_e^u)$

10.2 Cost of Debt

Case I: Cost of debt equal to the risk free rate and constant: $r = r_f$.

Case II: Cost of debt increases with the proportion of debt in the firm's capital structure.

10.3 Cost of Capital

Method 1: CAPM

$$WACC = \frac{V_d}{(V_d + V_e)} r + \frac{V_e}{(V_d + V_e)} E(\tilde{r}_e) \quad (\text{Without taxation})$$

$$WACC = \frac{V_d}{(V_d + V_e)} r(1 - \tau) + \frac{V_e}{(V_d + V_e)} E(\tilde{r}_e) \quad (\text{With taxation})$$

Method 2: Modigliani & Miller

$$WACC = E(\tilde{r}_0) = E(\tilde{r}_e^u) \quad (\text{Without taxation})$$

$$WACC = E(\tilde{r}_0) \left[1 - \tau \frac{V_d}{(V_d + V_e)} \right] \quad (\text{With taxation})$$

with $E(\tilde{r}_0) = E(\tilde{r}_e^u)$

10.4 Example

Consider the following historical information on ROMY.

- $\beta_e^l = 0.94$
- Ratio $V_d/V = 10\%$
- Treasury Bill rate (risk free) = 2%
- Market (equity) premium = 6%
- Bonds (10 years maturity) are rated AA
- Corporate tax rate = 30%
- Capital expenditures / year = 2,5% of sales
- Depreciation / year = 2,5% of sales
- WCR / year = 10% of sales
- Operating costs = 75% of sales
- Expected growth rate in sales: 10% for $t = 1 - 3$
- Expected growth rate in sales: 8% for $t = 4 - 7$
- Expected growth rate in sales: 3% for $t > 7$

Income Statement for ROMY (t) (000):

Sales	20 000
- Operating costs	15 000
EBITDA	5 000
- Depreciation	500
EBIT	4 500
- Interests	50
Taxable income	4 450
- Taxes	1 335
Net income	3 115

Case I: Interest rate equals to the risk free rate $r = r_f$ and no taxes

Step 1: Cash flows calculation (millions) for ROMY

Case 1: No taxes and risk-free debt

Calculations of FCF		Historical			Cash Flow Projections					TV
		0	1	2	3	4	5	6	7	7
Sales		20 000,0	22 000,0	24 200,0	26 620,0	28 749,6	31 049,6	33 533,5	36 216,2	
	Growth rate in sales		10%	10%	10%	8%	8%	8%	8%	
- Operating expenses		15 000,0	16 500,0	18 150,0	19 965,0	21 562,2	23 287,2	25 150,2	27 162,2	
EBITDA		5 000,0	5 500,0	6 050,0	6 655,0	7 187,4	7 762,4	8 383,4	9 054,1	
- Depreciation and amortization		500,0	550,0	605,0	665,5	718,7	776,2	838,3	905,4	
EBIT		4 500,0	4 950,0	5 445,0	5 989,5	6 468,7	6 986,2	7 545,0	8 148,6	
Taxes on EBIT		0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	
EBIT after tax		4 500,0	4 950,0	5 445,0	5 989,5	6 468,7	6 986,2	7 545,0	8 148,6	
+ Depreciation		500,0	550,0	605,0	665,5	718,7	776,2	838,3	905,4	
- CAPEX		500,0	550,0	605,0	665,5	718,7	776,2	838,3	905,4	
WCR		2 000,0	2 200,0	2 420,0	2 662,0	2 875,0	3 105,0	3 353,4	3 621,6	
Δ WCR		-	200,0	220,0	242,0	213,0	230,0	248,4	268,3	
FCF			4 750,0	5 225,0	5 747,5	6 255,7	6 756,2	7 296,6	7 880,4	

Step 2: Calculation of the firm's value

The value of a firm is equal to the present value of the Free Cash Flows. Thus, we can write:

$$V = \sum_{t=1}^7 \frac{CF_t}{(1+WACC)^t} + \frac{TV_7}{(1+WACC)^7}$$

with $TV_7 = \frac{CF_7(1+g)}{WACC-g}$

Method 1: CAPM

Calculation of β^u :

$$\beta^u = \frac{\beta_e^l}{\left[1 + \frac{V_d}{V_e}\right]} = \frac{0,94}{[1 + 0,111]} = 0.846$$

Therefore,

$$E(\tilde{r}_e^u) = r_f + \beta^u[E(\tilde{r}_m) - r_f] = 0.02 + 0.846(0.06) = 0.0708$$

Method 2: Modigliani & Miller

Cost of equity:

$$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e}[E(\tilde{r}_0) - r_f] = 0.0708 + \frac{V_d}{V_e}[0.0708 - 0.02]$$

with $E(r_0) = E(r_e^u) = 0.0708$

$$WACC = E(\tilde{r}_0) = E(r_e^u) = 0.0708$$

Calculation of the firm's value as a function of the capital structure.

Case 1: No taxes and risk-free debt

CAPM

V_d / V	V_d / V_e	R_f	$E(R_M) - R_f$	Tax rate	β^U	β^L	$E(R_e)$	R_d	WACC	Firm Value
0	0,000	0,020	0,0600	0,00	0,846	0,846	0,0708	0,020	0,0708	156 355,7
0,1	0,111	0,020	0,0600	0,00	0,846	0,940	0,0764	0,020	0,0708	156 355,7
0,2	0,250	0,020	0,0600	0,00	0,846	1,058	0,0835	0,020	0,0708	156 355,7
0,3	0,429	0,020	0,0600	0,00	0,846	1,209	0,0925	0,020	0,0708	156 355,7
0,4	0,667	0,020	0,0600	0,00	0,846	1,410	0,1046	0,020	0,0708	156 355,7
0,5	1,000	0,020	0,0600	0,00	0,846	1,692	0,1215	0,020	0,0708	156 355,7
0,6	1,500	0,020	0,0600	0,00	0,846	2,115	0,1469	0,020	0,0708	156 355,7
0,7	2,333	0,020	0,0600	0,00	0,846	2,820	0,1892	0,020	0,0708	156 355,7
0,8	4,000	0,020	0,0600	0,00	0,846	4,230	0,2738	0,020	0,0708	156 355,7

M&M

V_d / V	V_d / V_e	Tax rate	$E(R_0)$	R_f	$E(R_e)$	WACC	Firm Value
0	0,000	0,0	0,0708	0,02	0,0708	0,0708	156 355,7
0,1	0,111	0,0	0,0708	0,02	0,0764	0,0708	156 355,7
0,2	0,250	0,0	0,0708	0,02	0,0835	0,0708	156 355,7
0,3	0,429	0,0	0,0708	0,02	0,0925	0,0708	156 355,7
0,4	0,667	0,0	0,0708	0,02	0,1046	0,0708	156 355,7
0,5	1,000	0,0	0,0708	0,02	0,1215	0,0708	156 355,7
0,6	1,500	0,0	0,0708	0,02	0,1469	0,0708	156 355,7
0,7	2,333	0,0	0,0708	0,02	0,1892	0,0708	156 355,7
0,8	4,000	0,0	0,0708	0,02	0,2738	0,0708	156 355,7

Conclusions:

- If the cost of debt financing is equal to the risk free rate and there are no taxes, the CAPM and Modigliani & Miller approaches are equivalent.
- The cost of equity increases with the proportion of debt to reflect the increase in risk.
- The WACC is independent of the firm's capital structure
- The value of the firm is also independent of the capital structure.

Case II: Cost of debt is equal to the risk free rate ($r = r_f$) and taxes.

Case 2: Taxes and risk-free debt

Calculations of FCF	Historical	Cash Flow Projections							TV
	0	1	2	3	4	5	6	7	7
Sales	20 000,0	22 000,0	24 200,0	26 620,0	28 749,6	31 049,6	33 533,5	36 216,2	
<i>Growth rate in sales</i>		10%	10%	10%	8%	8%	8%	8%	
- Operating expenses	15 000,0	16 500,0	18 150,0	19 965,0	21 562,2	23 287,2	25 150,2	27 162,2	
EBITDA	5 000,0	5 500,0	6 050,0	6 655,0	7 187,4	7 762,4	8 383,4	9 054,1	
- Depreciation and amortization	500,0	550,0	605,0	665,5	718,7	776,2	838,3	905,4	
EBIT	4 500,0	4 950,0	5 445,0	5 989,5	6 468,7	6 986,2	7 545,0	8 148,6	
Taxes on EBIT	1 350,0	1 485,0	1 633,5	1 796,9	1 940,6	2 095,8	2 263,5	2 444,6	
EBIT after tax	3 150,0	3 465,0	3 811,5	4 192,7	4 528,1	4 890,3	5 281,5	5 704,1	
+ Depreciation	500,0	550,0	605,0	665,5	718,7	776,2	838,3	905,4	
- CAPEX	500,0	550,0	605,0	665,5	718,7	776,2	838,3	905,4	
WCR	2 000,0	2 200,0	2 420,0	2 662,0	2 875,0	3 105,0	3 353,4	3 621,6	
Δ WCR	-	200,0	220,0	242,0	213,0	230,0	248,4	268,3	
FCF		3 265,0	3 591,5	3 950,7	4 315,1	4 660,3	5 033,1	5 435,8	

Step 2: Calculation of the firm's value

Method 1: CAPM

Calculation of β^u :

$$\beta^u = \frac{\beta_e^l}{\left[1 + (1 - \tau)\frac{V_d}{V_e}\right]} = \frac{0.94}{[1 + (1 - 0.3)(0.111)]} = 0.872$$

Therefore,

$$E(\tilde{r}_e^u) = r_f + \beta^u[E(\tilde{r}_m) - r_f] = 0.02 + 0.872(0.06) = 0.0723$$

Method 2: Modigliani & Miller

Calculation of the cost of equity:

$$E(\tilde{r}_e) = E(\tilde{r}_0) + \frac{V_d}{V_e}[E(\tilde{r}_0) - r_f](1 - \tau) = 0.0723 + \frac{V_d}{V_e}[0.0723 - 0.02](1 - 0.3)$$

Calculation of the WACC:

$$CMPC = E(\tilde{r}_0) \left[1 - \tau \frac{V_d}{(V_d + V_e)}\right] = 0.0723 \left[1 - 0.3 \frac{V_d}{(V_d + V_e)}\right]$$

CAPM											
	Vd / V	Vd / Ve	Rf	E(Rm) - Rf	Tax rate	β^u	β^L	E(Re)	Rd	WACC	Firm Value
	0	0,000	0,020	0,0600	0,30	0,872	0,872	0,0723	0,020	0,0723	103 691,3
	0,1	0,111	0,020	0,0600	0,30	0,872	0,940	0,0764	0,020	0,0702	109 482,2
	0,2	0,250	0,020	0,0600	0,30	0,872	1,025	0,0815	0,020	0,0680	115 936,6
	0,3	0,429	0,020	0,0600	0,30	0,872	1,134	0,0880	0,020	0,0658	123 175,3
	0,4	0,667	0,020	0,0600	0,30	0,872	1,279	0,0968	0,020	0,0637	131 350,0
	0,5	1,000	0,020	0,0600	0,30	0,872	1,483	0,1090	0,020	0,0615	140 654,2
	0,6	1,500	0,020	0,0600	0,30	0,872	1,788	0,1273	0,020	0,0593	151 339,0
	0,7	2,333	0,020	0,0600	0,30	0,872	2,297	0,1578	0,020	0,0571	163 735,4
	0,8	4,000	0,020	0,0600	0,30	0,872	3,314	0,2189	0,020	0,0550	178 289,7

M&M								
	Vd / V	Vd / Ve	Tax rate	E(R0)	Rf	E(Re)	WACC	Firm Value
	0	0,000	0,3	0,0723	0,02	0,0723	0,0723	103 691,3
	0,1	0,111	0,3	0,0723	0,02	0,0764	0,0702	109 482,2
	0,2	0,250	0,3	0,0723	0,02	0,0815	0,0680	115 936,6
	0,3	0,429	0,3	0,0723	0,02	0,0880	0,0658	123 175,3
	0,4	0,667	0,3	0,0723	0,02	0,0968	0,0637	131 350,0
	0,5	1,000	0,3	0,0723	0,02	0,1090	0,0615	140 654,2
	0,6	1,500	0,3	0,0723	0,02	0,1273	0,0593	151 339,0
	0,7	2,333	0,3	0,0723	0,02	0,1578	0,0571	163 735,4
	0,8	4,000	0,3	0,0723	0,02	0,2189	0,0550	178 289,7

Conclusions:

- If the cost of debt financing is equal to the risk free rate and there are taxes, the CAPM and Modigliani & Miller approaches are equivalent.
- The cost of equity increases with the proportion of debt to reflect the increase in risk.
- When the tax rate is positive but the cost of debt does not vary with the proportion of debt, the WACC decreases with leverage.
- The value of the firm increases with leverage.

Problem: An increase in leverage will have an impact on the cost of debt financing since it increases the default probability.

Case III: Cost of debt increases with the proportion of debt in the firm's capital structure and taxes.

The cost of debt financing varies according to the firm's default probability corresponding to different level of leverage. The default probability is reflected in the "spread" which indicates the risk premium over the Treasury Bill rate. Suppose the following spread structure for 10 years maturity bonds. The spread are expressed in % points. Consider the following spreads:

$V_d/(V_d + V_e)$	Spread (S) Basis point
0.0	30
0.1	50
0.2	65
0.3	88
0.4	125
0.5	170
0.6	290
0.7	400
0.8	600

Note: This section relies on the CAPM to calculate the cost of equity and the WACC.

Calculation of the firm's value as a function of the capital structure.

Case 3: Taxes and risky debt

CAPM

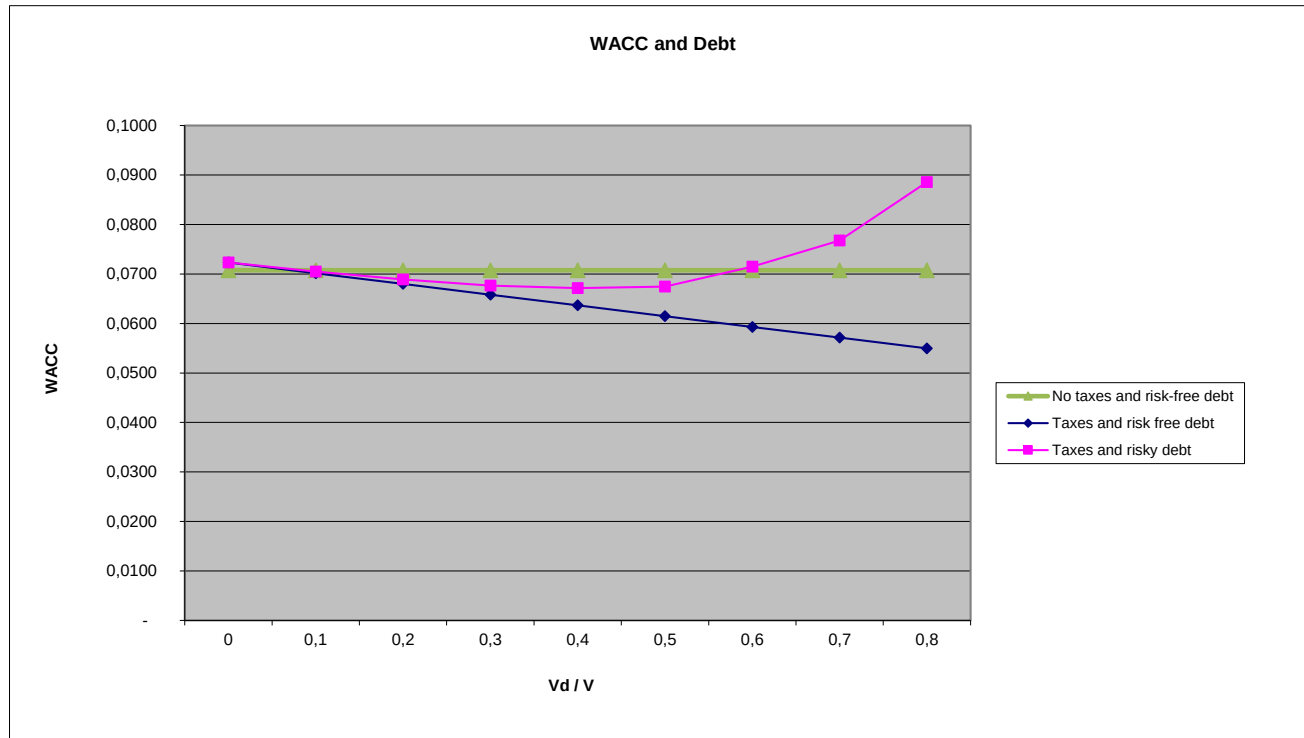
V_d / V	V_d / V_e	R_f	$E(R_M) - R_f$	Tax rate	β^u	β^L	$E(R_e)$	Spread	R_d	WACC	Firm Value
0	0,000	0,020	0,0600	0,30	0,872	0,872	0,0723	0,0030	0,0230	0,0723	103 691,3
0,1	0,111	0,020	0,0600	0,30	0,872	0,940	0,0764	0,0050	0,0250	0,0705	108 506,0
0,2	0,250	0,020	0,0600	0,30	0,872	1,025	0,0815	0,0065	0,0265	0,0689	113 141,8
0,3	0,429	0,020	0,0600	0,30	0,872	1,134	0,0880	0,0088	0,0288	0,0677	116 957,6
0,4	0,667	0,020	0,0600	0,30	0,872	1,279	0,0968	0,0125	0,0325	0,0672	118 637,5
0,5	1,000	0,020	0,0600	0,30	0,872	1,483	0,1090	0,0170	0,0370	0,0674	117 723,2
0,6	1,500	0,020	0,0600	0,30	0,872	1,788	0,1273	0,0290	0,0490	0,0715	105 859,3
0,7	2,333	0,020	0,0600	0,30	0,872	2,297	0,1578	0,0400	0,0600	0,0767	93 582,8
0,8	4,000	0,020	0,0600	0,30	0,872	3,314	0,2189	0,0600	0,0800	0,0886	74 014,8

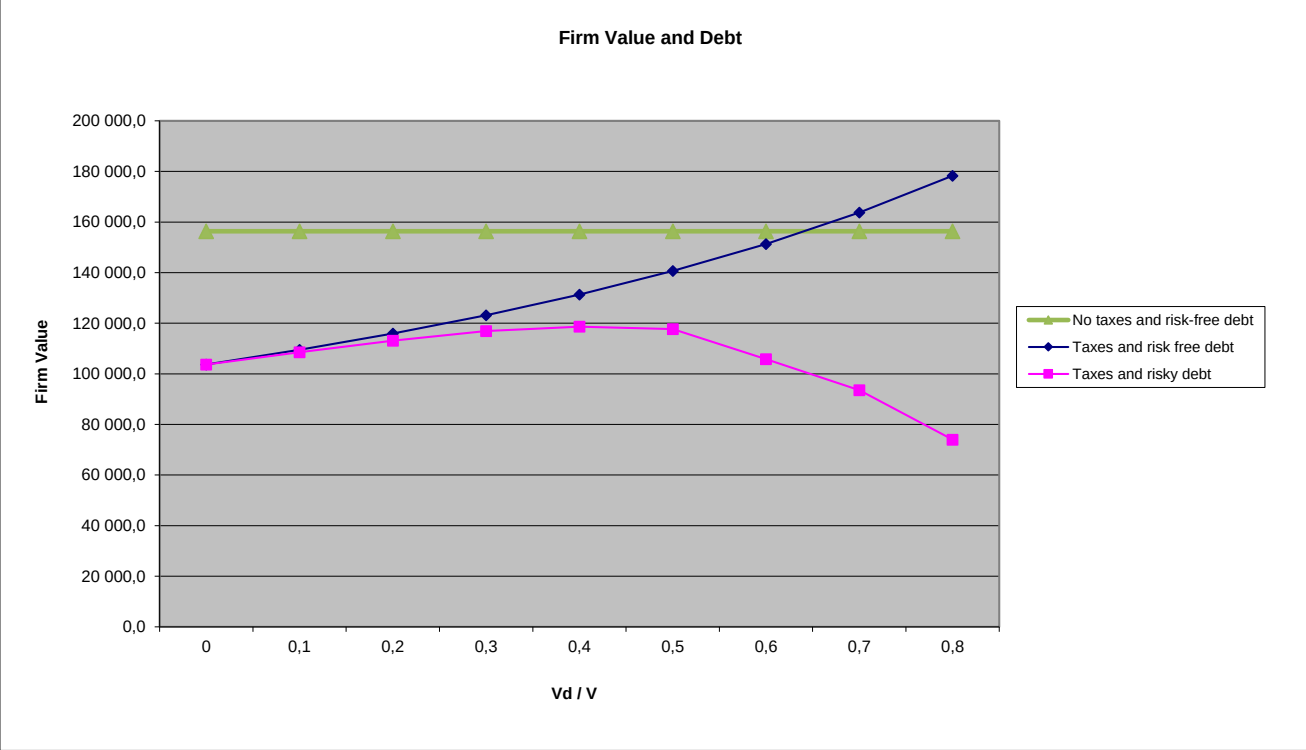
Assumption: The formula for the levered beta does not take into account the fact that the debt is risky (Beta of debt greater than zero)

Conclusions:

- The WACC first decreases with leverage and then increases to follow the increase in risk.
- The WACC reaches its minimum value at 6.74% when the proportion of debt reaches 50%.
- The firm's value is maximum at 117 723.2 when the WACC reaches its minimum.

Note: This approach is similar to the Trade-Off approach. The increase in the cost of debt financing is used to capture the increase in the probability of bankruptcy.





10.5 Conflict of interests: shareholders vs. creditors

There are two general types of investors in a firm:

1. shareholders (equity)
2. creditors (debt)

These investors have different interests and may have different incentives on the firm's investment policy.

Consider the following information on LOULOU:

- Assets
 - Cash: 10M
 - Fixed assets: Can take 2 values at $T = 1$:
 - * 49.7M (High state)
 - * 24.7M (Low state)
 - * prob (each state) : 50%
- Debt
 - Debt : 45M to be reimbursed at $T = 1$
 - Interests on debt: 5M to be paid at $T = 1$

Other information:

- The firm plans to cease activity at $T = 1$ and can invest the 10M of cash in two different projects.
- **Project 1:** Invest 10M in a risk-free bond with an annual return of 3%.
- **Project 2:** Invest 10M in a risky project which can generate the following cash flows at $T = 1$:
 - 16M (High state)
 - 0M (Low state)
 - prob (each state) : 50%
- At $T = 1$, the company is liquidated and the proceeds distributed to investors.

Question: Which project should be undertaken by the firm?

Answer: Consider the following cash flows to creditors and shareholders at $T = 1$ for both projects.

1. Project 1

- The firm invests 10M in a risk-free bond with an annual return of 3%.
- The bond pays 300 000 of interests at $T = 1$.

The cash flows in each state of nature at $T = 1$ are:

High state

Assets		Liabilities	
Cash	10.3	Debt	50
Fixed assets	49.7	Equity	10
Total	60	Total	60

Low state

Assets		Liabilities	
Cash	10.3	Debt	35
Fixed assets	24.7	Equity	0
Total	35	Total	35

2. Project 2

- The firm invests in a risky project which pays either 16M or 0M with 50% probability.
- For simplicity, let's assume a zero discount rate. The NPV of the project is equal to:

$$NPV = -10 + 0.5(16+0) = -2$$
- Conclusion: this project destroys value (also true for $r > 0$).

The cash flows in each state of nature at $T = 1$ are:

High state

Assets		Liabilities	
Project	16	Debt	50
Fixed assets	49.7	Equity	15.7
Total	65.7	Total	65.7

Low state

Assets		Liabilities	
Project	0	Debt	24.7
Fixed assets	24.7	Equity	0
Total	24.7	Total	24.7

Now consider the cash flows to both types of investors in each state of nature.

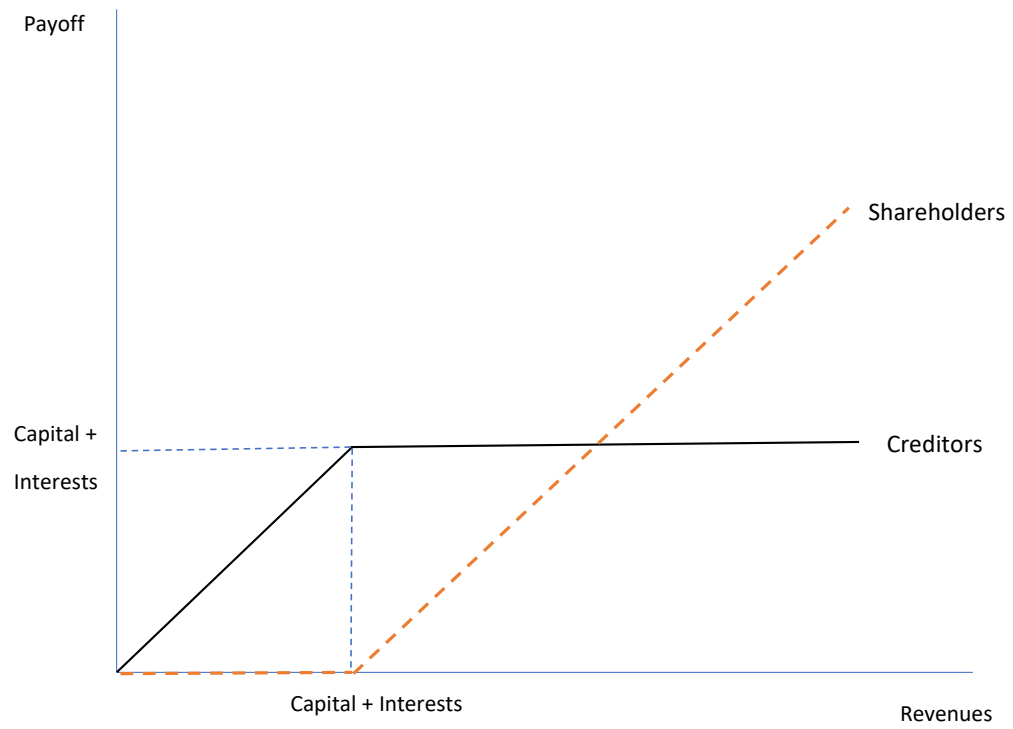
	Shareholders		Creditors	
	High state 1	Low state	High state	Low state
Project 1	10	0	50	35
Project 2	15.7	0	50	24.7

Explanations:

- Shareholders: In the low state, they receive 0 with both projects. Yet, in the High state, they receive a larger payoff with project 2 (risky) than with project 1 (risk-free).
- Creditors: In the high state, they receive 50 under both projects. However, in the low state they receive less with project 2 (risky) than with project 1 (risk-free).

Conclusions:

- Shareholders get all upside of the risky projects while they transfer the downside to creditors.
- This can be better understood with the graph which depicts the payoff to shareholders and creditors.
- A share represents a **call option** on the firm's cash flows with a strike equal to the debt and interests payment. The shareholders' payoff function is convex (risk loving).
- The creditors payoff function is concave (risk aversion)
- If the management's objective is to maximize the value of shareholders' equity, he will opt for the riskier project even if it destroys value.



Application 2 - Capital Structure (In class)

Capital structure decisions can be key decisions to be taken by a firm's manager. One important question at stake is whether or not financing decisions create value and if so, who get the benefits from these strategic decisions.

Generally speaking, we can write that the value of a firm is equal to:

$$V = \sum_{t=1}^{\infty} \frac{FCF}{(1 + WACC)^t}$$

where FCF represent the Free Cash Flows and $WACC$ is the weighted average cost of capital.

Therefore, if capital structure decisions impact on a firm's value, it must either be through its impact on i) the free (operational) cash flows (FCF) or ii) the discount rate (WACC).

In practice, there are different approaches to calculating the value of a firm. A first natural approach is to calculate the firm's value as the present value of the firm's free cash flows. This is referred to the Discounted Cash Flow (DCF) method.

A second approach to valuation considers the value of the firm as the sum of the cash flows to the different investors in the firm (i.e. creditors and equityholders).

Finally, a third approach to valuation is derived from the work of Modigliani & Miller and referred to as the Adjusted Present Value (APV) method whereby the value of a firm can be decomposed into:

- Value of operational cash flows of the unlevered firm
- PV (tax shield on debt)

In order to illustrate these three approaches and examine their consistency, consider the following information on ROMY Inc.

- Debt is a perpetuity and pays interests every year with no capital repayment
- The pretax cost of debt is equal to the risk-free rate of 3%
- EBIT: 5 000 000 per year at perpetuity
- Depreciation & amortization (D&A): 750 000 per year at perpetuity
- Investment expenses (CAPEX): 750 000 per year at perpetuity
- No annual variation in WCR
- Corporate tax rate: 30%
- Unlevered equity beta of ROMY: 1
- Expected return of the market: 10%
- The share price of the unlevered firm is 14

DCF

In a classic DCF approach, Free Cash Flows are independent of the firm's capital structure since we focus on the firm's cash flows irrespective of who hold specific claims against these cash flows. Indeed, we can write:

$$FCF = EBITDA(1 - \tau) + \tau \times D\&A - CAPEX - \Delta WCR$$

or

$$FCF = EBIT(1 - \tau) + D\&A - CAPEX - \Delta WCR$$

Thus, in the DCF approach, the effect of capital structure decisions is captured by the Weighted Average Cost of Capital (WACC).

The cost of equity and the WACC can be calculated from two methods: i) CAPM and ii) Modigliani & Miller.

- Cost of equity financing
 - CAPM: $E(r_e) = r_f + \beta_e [E(r_m) - r_f]$
 - M&M: $E(r_e) = E(r_e^u) + V_d/V_e [E(r_e^u) - r_f](1 - \tau)$
- WACC
 - CAPM: $WACC = \frac{V_d}{V} r_f(1 - \tau) + \frac{V_e}{V} E(r_e)$
 - M&M: $WACC = E(r_e^u)[1 - \tau V_d/V]$

In addition, we have the relationship between the beta of an unlevered firm, β^u , and its beta if it was levered, β_e^l .

$$\beta_e^l = \left[1 + (1 - \tau) \frac{V_d}{V_e} \right] \beta^u$$

Complete Table 1 for three different capital structures of ROMY Inc.

Table 1

	A	B	C
Market value of debt	0	4 000 000	12 000 000
Market value of equity	35 000 000	32 200 000	26 600 000
Total market value	35 000 000	36 200 000	38 600 000
% Debt (market value)	—	—	—
% Equity (market value)	—	—	—
Free Cash Flows (FCF)	—	—	—
Cost of debt (Risk-free rate)	3.00%	3.00%	3.00%
After-tax cost of debt	—	—	—
Unlevered beta	1.00	1.00	1.00
Cost of unlevered equity	—	—	—
Levered beta	—	—	—
Cost of levered equity (CAPM)	—	—	—
Cost of levered equity (M&M)	—	—	—
WACC (CAPM)	—	—	—
WACC (M&M)	—	—	—
Enterprise value	—	—	—
Market value of equity	—	—	—

Cash flows to investors

An alternative approach is to consider the cash flows to the different investors in the firm (i.e. creditors and equityholders). In the case of perpetual debt, the annual cash flows to creditors are given by the interest payments equal to:

$$I = r_f \times V_d$$

As residual claimants, equityholders are entitled to receive all the cash flows after everything has been paid. This corresponds to equity cash flows and are equal to :

$$ECF = (EBIT - I)(1 - \tau) + D\&A - CAPEX - \Delta WCR - RLTD$$

where *RLTD* is equal to the repayment of long term debt. This is equal to zero in the case of perpetual debt.

Complete Table 2 for three different capital structures of ROMY Inc.

Table 2

	A	B	C
Creditors			
Market value of debt	0	4 000 000	12 000 000
Pretax cost of debt	3.00%	3.00%	3.00%
Interests on debt	—	—	—
Equityholders			
Equity cash flows (ECF)	—	—	—
Cost of equity (Table 1)	—	—	—
Market value of equity	—	—	—
Enterprise value	—	—	—

APV

Another approach to firm valuation is the Adjusted Present Value (APV) whereby the value of a firm is equal to:

$$V^l = V^u + \tau V_d$$

Note: We assume that bankruptcy costs are zero.

In this setting, the value of a firm can be decomposed into:

- Value of operational cash flows of the unlevered firm: $V^u = FCF/E(r_e^u)$
- PV (tax shield on debt): τV_d

Complete Table 3 for three different capital structures of ROMY Inc.

Table 3

	A	B	C
FCF	3 500 000	3 500 000	3 500 000
Cost of unlevered equity (Table 1)	—	—	—
V^u	—	—	—
PV (tax shield)	—	—	—
Enterprise value (V^l)	—	—	—

Gain to shareholders

As we have seen in the previous section, leverage allows a firm to benefit from a tax shield since interests are tax deductible. This benefit is captured by equityholders since they are the residual claimants. Yet, the underlying process by which equityholders benefit may not be obvious. Indeed, it is assumed that the total amount of debt issued by the firm is not used for investment purposes but rather used to repurchase shares on the market. Thus, the issue of new debt does not increase the firm's cash flows (EBIT is assumed to be constant).

Given the one for one exchange of debt and equity, we can anticipate that the new market value of equity in the firm will decrease following the share repurchase. But what about the share price?

To understand what happens to the individual share price, we need to decompose this process into the following sequence:

1. Determine the actual share price and the number of shares before the issue of new debt and share repurchase
2. Calculate the total value of the tax shield generated by the issue of new debt
3. Calculate the tax shield per share captured by the existing equityholders
4. Calculate the new share price equal to the old share price plus the tax shield per share
5. Calculate the number of shares repurchased with the new debt. This is equal to the value of debt issued divided by the new share price
6. Calculate the number of shares remaining on the market
7. Verify that the value of equity after the share repurchase is equal to the new share price times the number of shares after the repurchase

Complete Table 4 for three different capital structures of ROMY Inc.

Table 4

	A	B	C
Mkt value of equity	35 000 000	32 200 000	26 600 000
Nbr of shares (before leverage)	2 500 000	2 500 000	2 500 000
Share price (before leverage)	14	14	14
Mkt value of debt	0	4 000 000	12 000 000
PV (tax shield)	—	—	—
PV (tax shield) per share	—	—	—
Share price (after leverage)	—	—	—
Nbr of share repurchased	—	—	—
Nbr of share after repurchase	—	—	—
Mkt value of equity after repurchase	—	—	—
Enterprise value after repurchase	—	—	—

September 10, 2009
Pfizer Inc

VALUATION

Discounted cash flow

DCF-based price target reflects revenue loss and pipeline uncertainty in the out-years.

Assuming a WACC of 8% (we assume a higher rate than the 7% according to Bloomberg because of debt payment), we derive a DCF value of \$19 assuming a -3% growth rate in perpetuity, given that long-term R&D productivity remains uncertain. We expect the company to generate \$111B in free cash flow over the next six years (2010–2015), which is 85% of pro-forma market cap of \$130B.

and used a terminal growth rate of -8%. We note that this is a highly theoretical analysis and that small changes in the terminal growth rate cause significant changes in the resulting valuation. Please contact our office if you would like a copy of this separate analysis.

\$19 target represents 2010E P/E of 8.5, which is 37% discount to the S&P multiple.

Pfizer's P/E on 2010E is 8.5x, which is a 37% discount to the S&P's multiple of 13.4x (using S&P level of 994.75 divided by EPS of \$74.10).

We calculate that PFE is worth \$14 excluding the pipeline. In our analysis, we started with our existing income statement, removed pipeline revenue (\$6.3B in 2015), reduced spending,

Exhibit 8

Pfizer DCF Valuation

[illegible]

*We apply higher discount rate than WACC of 7% (Bloomberg) because we assume debt payment.

*We assume one-time charge expense in out years for cash restructuring and litigation outflows.

Source: Company data, Morgan Stanley Research

Valuation

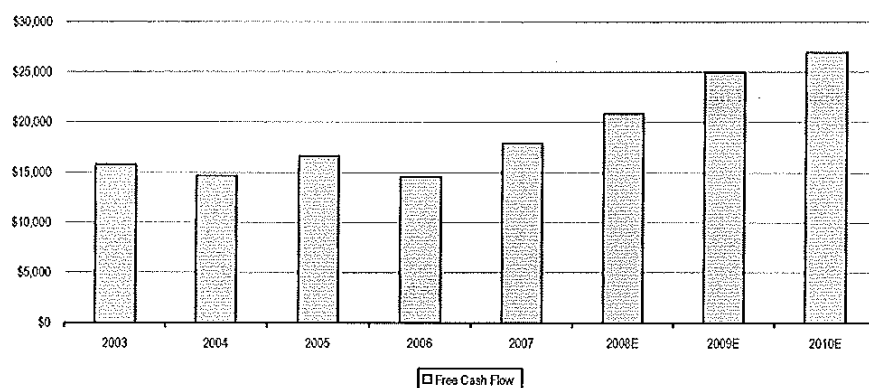
To provide an in-depth valuation for Microsoft, we have run discounted cash flow, implied market expectations, and relative valuation analyses for Microsoft. Our discounted cash flow analysis drives to a stock price in the high \$30s. A \$35 target price implies P/E multiples of 16.1 and 13.8 times our NTM and fiscal 2010 estimates, respectively, versus 11.7 times NTM and 9.9 times fiscal 2010 P/E multiples, based on yesterday's closing stock price.

Discounted Cash Flow Analysis

We have valued Microsoft using a discounted cash flow (DCF) analysis, which justifies an intrinsic share price of \$38.00 per share. We forecast Microsoft's revenue to grow 10.9% from fiscal 2009 to fiscal 2010, and we forecast free cash flow growing from \$21.49 billion in fiscal 2009 to \$22.98 billion in fiscal 2010, implying an enterprise value to free cash flow multiple of 8.8 times based on yesterday's closing stock price.

Exhibit 24: Free Cash Flow Analysis

US\$ in millions, unless otherwise stated



Source: Company data, Credit Suisse estimates.

Sum-of-the-Parts Discounted Cash Flow Analysis

We also performed a sum-of-the-parts analysis on Microsoft utilizing a straightforward ten-year discounted cash flow analysis based on our estimates, breaking out the company's cash flow contributions for each division.

Exhibit 25: Free Cash Flow Growth Rates by Division

	10-Year CAGR (%)
Client	8.7%
Server & Tools	10.5%
Online Services Business	NM
Microsoft Business Division	7.0%
Entertainment and Devices	3.5%
Total	8.7%

Source: Company data, Credit Suisse estimates.

For the company as a whole, we forecast free cash flow to grow at an 8.7% compounded annual growth rate from fiscal 2008 through fiscal 2018. Of this combined growth, we estimate that the Server & Tools division will experience the highest growth, while Microsoft's two largest divisions, Client and Microsoft Business Division, will still show healthy high single-digits growth in free cash flow. (See Exhibit 25.)

Structure du capital

MM and the Real World

Students often question why Modigliani and Miller's results are important if, after all, capital markets are not perfect in the real world. While it is true that capital markets are not perfect, *all* scientific theories begin with a set of idealized assumptions from which conclusions can be drawn. When we apply the theory, we must then evaluate how closely the assumptions hold, and consider the consequences of any important deviations.

As a useful analogy, consider Galileo's law of falling bodies. Galileo overturned the conventional wisdom by showing that, without friction, free-falling bodies will fall at the same rate independent of their mass. If you test this law, you will likely find it does not hold exactly. The

reason, of course, is that unless we are in a vacuum, air friction tends to slow some objects more than others.

MM's results are similar. In practice, we will find that capital structure can have an effect on firm value. But just as Galileo's law of falling bodies reveals that we must look to air friction, rather than any underlying property of gravity, to explain differences in the speeds of falling objects, MM's proposition reveals that any effects of capital structure must similarly be due to frictions that exist in capital markets. After exploring the full meaning of MM's results in this chapter, we look at the important sources of these frictions, and their consequences, in subsequent chapters.

Source: Berk & de Nayer

Microsoft's Dividend, Cash, and Beta

In mid-2004, Microsoft had a market capitalization of more than \$300 billion and nearly \$60 billion in cash and short-term investments, with no debt outstanding. In November 2004, the firm used some of its cash to pay a one-time dividend of \$32 billion. By paying out this cash,

Microsoft *increased* its net debt from -\$60 billion to -\$28 billion. The effect of the dividend is equivalent to an increase in leverage: Paying out the cash will cause the beta of Microsoft's stock to be higher than it otherwise would be.

Chapter 14 Capital Structure in a Perfect Market

structure de capital

Nobel Prize Franco Modigliani and Merton Miller

Franco Modigliani and Merton Miller, the authors of the Modigliani-Miller Propositions, have each won the Nobel Prize in economics for their work in financial economics, including their capital structure propositions. Modigliani won the Nobel Prize in 1985 for his work on personal savings and for his capital structure theorems with Miller. Miller earned his prize in 1990 for his analysis of portfolio theory and capital structure.

Miller once described the MM propositions in an interview this way:

People often ask: Can you summarize your theory quickly? Well, I say, you understand the M&M theorem if you know why this is a joke: The pizza delivery man comes to Yogi Berra after the game and says, "Yogi, how do you want this pizza cut, into quarters or eighths?" And Yogi says, "Cut it in eight pieces. I'm feeling hungry tonight."

*Everyone recognizes that's a joke because obviously the number and shape of the pieces don't affect the size of the pizza. And similarly, the stocks, bonds, warrants, et cetera, issued don't affect the aggregate value of the firm. They just slice up the underlying earnings in different ways.**

Modigliani and Miller each won the Nobel Prize in large part for their observation that the value of a firm should be unaffected by its capital structure in perfect capital markets. While the intuition underlying the MM propositions may be as simple as slicing pizza, their implications for corporate finance are far-reaching. The propositions imply that the true role of a firm's financial policy is to deal with (and potentially exploit) financial market imperfections such as taxes and transaction costs. Modigliani and Miller's work began a long line of research into these market imperfections, which we look at over the next several chapters.

*Peter J. Tanous, *Investment Gurus* (New York: Institute of Finance, 1997).

Some: Berk: De Nango

Cost of Capital

TABLE 8.3 Current Practices for Estimating Cost of Capital

Cost of Capital Item	Current Practices
Cost of equity	• 81% of firms used the capital asset pricing model to estimate the cost of equity, 4% used a modified capital asset pricing model, and 15% were uncertain about how they estimated the cost of equity. • 70% of firms used 10-year Treasuries or longer as the riskless rate, 7% used 3- to 5-year Treasuries, and 4% used the Treasury bill rate. • 52% used a published source for a beta estimate, while 30% estimated it themselves. • There was wide variation in the market risk premium used, with 37% using a premium between 5% and 6%.
Cost of debt	• 52% of firms used a marginal borrowing rate and a marginal tax rate, while 37% used the current average borrowing rate and the effective tax rate.
Weights for debt and equity	• 59% used market value weights for debt and equity in the cost of capital, 15% used book value weights, and 19% were uncertain about what weights they used.

Source: Bruner, Eades, Harris, and Higgins (1998).

11 Mathematical Refresher

11.1 Derivatives

i) Constant function: $y = f(x) = a$

$$\frac{dy}{dx} = 0$$

ii) Power Function: $y = f(x) = ax^n$

$$\frac{dy}{dx} = anx^{n-1}$$

iii) Logarithmic function: $y = f(x) = a \ln(x)$

$$\frac{dy}{dx} = \frac{a}{x}$$

iv) Exponential function: $y = f(x) = e^{ax}$

$$\frac{dy}{dx} = ae^{ax}$$

v) Derivative of a sum or a subtraction: $y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

vi) Derivative of a product: $y = f(x)g(x)$

$$\frac{dy}{dx} = \frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x) = g(x)f'(x) + f(x)g'(x)$$

vii) Derivative of a division: $y = f(x)/g(x)$

$$\frac{dy}{dx} = \frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

viii) Chain rule: $z = f(y)$ with $y = g(x)$

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = f'(y) g'(x)$$

11.2 Integrals

Integration represents the opposite operation to differentiation (derivatives). Let $F(x)$ be a function and $f(x)$ its derivative. Then we can write:

$$\frac{dF(x)}{dx} = f(x) \Rightarrow \int f(x)dx = F(x) + c \quad (c = \text{constant of integration})$$

Indefinite integrals

i) Constant function

$$\int a \, dx = ax + c$$

ii) Power function

$$\int ax^n \, dx = \frac{a}{n+1} x^{n+1} + c \quad (n \neq -1)$$

We can verify that the integral is the inverse operation to the derivative. Indeed:

$$\frac{dF(x)}{dx} = \frac{d}{dx} \left(\frac{ax^{n+1}}{n+1} \right) = ax^n = f(x)$$

iii) Exponential function

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

iv) Logarithmic function

$$\int \frac{a}{x} \, dx = a \ln(x) + c \quad (x > 0)$$

v) Integral of a sum of a subtraction

$$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

vi) Integral of a product by a constant

$$\int af(x) \, dx = a \int f(x) \, dx$$

Definite integrals:

$$\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a)$$

Ex: Solve the following integral:

$$\int_1^5 3x^2 \, dx = \frac{3}{3} x^3 \Big|_1^5 = x^3 \Big|_1^5 = (5)^3 - (1)^3 = 124$$

11.3 Matrix notation

Def: A **matrix** is a rectangular array of numbers with m rows and n columns. The rows and the columns of the matrix define its dimension, that is $m \times n$.

Example: Matrix A of dimension 4×4 :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

i) Addition and subtraction of matrices

Two matrices can be added or subtracted if and only if they have the same dimension. Let A and B be two matrices. The sum of A et B is a new matrix defined as the addition of each pair of corresponding elements.

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Example:

$$A = \begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix} \quad A + B = \begin{bmatrix} -1 & 1 \\ -1 & 8 \end{bmatrix}$$

ii) Multiplication of matrices

To multiply two matrices, the column dimension of the first matrix must be equal to the row dimension of the second matrix. Note: The product of AB is generally not equal to the product of BA .

$$A_{(m \times p)} B_{(p \times n)} = C_{(m \times n)}$$

The element c_{ij} of matrix C is the sum of the multiplication of the elements of the i -th row of A with the elements of the j -th column of B .

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$$

Example:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}_{(2 \times 3)} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}_{(3 \times 3)}$$

A (2×3) matrix multiplying a (3×3) matrix will generate a (2×3) matrix:

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (1 \times 1) + (-1 \times 0) + (2 \times 1) = 3 \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = (1 \times 2) + (-1 \times -1) + (2 \times 2) = 7 \\ c_{13} &= a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = (1 \times 0) + (-1 \times 1) + (2 \times -1) = -3 \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = (3 \times 1) + (0 \times 0) + (1 \times 1) = 4 \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = (3 \times 2) + (0 \times -1) + (1 \times 2) = 8 \\ c_{23} &= a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = (3 \times 0) + (0 \times 1) + (1 \times -1) = -1 \end{aligned}$$

Hence,

$$C = \begin{bmatrix} 3 & 7 & -3 \\ 4 & 8 & -1 \end{bmatrix}_{(2 \times 3)}$$

Note: It is not possible to divide one matrix by another.

iii) Commutative, Associative and Distributive Laws

1. Commutative law: $A + B = B + A$
2. Associative law: $(A + B) + C = A + (B + C)$
3. Associative law: $(AB)C = A(BC) = ABC$
4. Distributive law: $A(B + C) = AB + AC$

Note: The condition for multiplication and addition of matrices must apply.

iv) The transpose of a matrix is obtained by interchanging its rows and columns, so the first row becomes the first column, etc. The new matrix is denoted A' or A^T . If A is of dimension $(m \times n)$, then A' will be of dimension $(n \times m)$.

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 1 & 2 \end{bmatrix}_{(3 \times 2)} \quad A' = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 0 & 2 \end{bmatrix}_{(2 \times 3)}$$

Def: A matrix is said to be **diagonal** if $a_{ij} = a_{ji}$ pour $i = j$ et $a_{ij} = 0$ pour $i \neq j$.

Def: A matrix is said to be **symmetric** if $a_{ij} = a_{ji} \quad \forall i, j$.

Def: A matrix is said to be **anti-symmetric** if $a_{ij} = -a_{ji} \quad \forall i, j$.

Def: A **vector** is defined as a matrix with one row and n columns. a matrix with one column and n rows is denoted a column vector.